

ω -powers and descriptive set theory.

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Abstract. We study the sets of the infinite sentences constructible with a dictionary over a finite alphabet, from the viewpoint of descriptive set theory. Among other things, this gives some true co-analytic sets. The case where the dictionary is finite is studied and gives a natural example of a set at the level ω of the Wadge hierarchy.

1 Introduction.

We consider the finite alphabet $n = \{0, \dots, n-1\}$, where $n \geq 2$ is an integer, and a dictionary over this alphabet, i.e., a subset A of the set $n^{<\omega}$ of finite words with letters in n .

Definition 1 *The ω -power associated to A is the set A^ω of the infinite sentences constructible with A by concatenation. So we have $A^\omega := \{a_0 a_1 \dots \in n^\omega / \forall i \in \omega \ a_i \in A\}$.*

The ω -powers play a crucial role in the characterization of subsets of n^ω accepted by finite automata (see Theorem 2.2 in [St1]). We will study these objects from the viewpoint of descriptive set theory. The reader should see [K1] for the classical results of this theory; we will also use the notation of this book. The questions we study are the following:

- (1) What are the possible levels of topological complexity for the ω -powers? This question was asked by P. Simonnet in [S], and studied in [St2]. O. Finkel (in [F1]) and A. Louveau proved independently that Σ_1^1 -complete ω -powers exist. O. Finkel proved in [F2] the existence of a Π_m^0 -complete ω -power for each integer $m \geq 1$.
- (2) What is the topological complexity of the set of dictionaries whose associated ω -power is of a given level of complexity? This question arises naturally when we look at the characterizations of Π_1^0 , Π_2^0 and Σ_1^0 ω -powers obtained in [St2] (see Corollary 14 and Lemmas 25, 26).
- (3) We will recall that an ω -power is an analytic subset of n^ω . What is the topological complexity of the set of codes for analytic sets which are ω -powers? This question was asked by A. Louveau. This question also makes sense for the set of codes for Σ_ξ^0 (resp., Π_ξ^0) sets which are ω -powers. And also for the set of codes for Borel sets which are ω -powers.

As usual with descriptive set theory, the point is not only the computation of topological complexities, but also the hope that these computations will lead to a better understanding of the studied objects. Many sets in this paper won't be clopen, in particular won't be recursive. This gives undecidability results.

- We give the answer to Question (2) for the very first levels ($\{\emptyset\}$, its dual class and Δ_1^0). This contains a study of the case where the dictionary is finite. In particular, we show that the set of dictionaries whose associated ω -power is generated by a dictionary with two words is a $\check{D}_\omega(\Sigma_1^0)$ -complete set. This is a surprising result because this complexity is not clear at all on the definition of the set.

- We give two proofs of the fact that the relation “ $\alpha \in A^\infty$ ” is Σ_1^1 -complete. One of these proofs is used later to give a partial answer to Question (2). To understand this answer, the reader should see [M] for the basic notions of effective descriptive set theory. Roughly speaking, a set is effectively Borel (resp., effectively Borel in A) if its construction based on basic clopen sets can be coded with a recursive (resp., recursive in A) sequence of integers. This answer is the

Theorem. *The following sets are true co-analytic sets:*

- $\{A \in 2^{n^{<\omega}} / A^\infty \in \Delta_1^1(A)\}$.
- $\{A \in 2^{n^{<\omega}} / A^\infty \in \Sigma_\xi^0 \cap \Delta_1^1(A)\}$, for $1 \leq \xi < \omega_1$.
- $\{A \in 2^{n^{<\omega}} / A^\infty \in \Pi_\xi^0 \cap \Delta_1^1(A)\}$, for $2 \leq \xi < \omega_1$.

This result also comes from an analysis of Borel ω -powers: A^∞ is Borel if and only if we can choose in a Borel way the decomposition of any sentence of A^∞ into words of A (see Lemma 13). This analysis is also related to Question (3) and to some Borel uniformization result for G_δ sets locally with Borel projections. We will specify these relations.

- A natural ordinal rank can be defined on the complement of any ω -power, and we study it; its knowledge gives an upper bound of the complexity of the ω -power.
- We study the link between Question (1) and the extension ordering on finite sequences of integers.
- Finally, we give some examples of ω -powers complete for the classes Δ_1^0 , $\Sigma_1^0 \oplus \Pi_1^0$, $D_2(\Sigma_1^0)$, $\check{D}_2(\Sigma_1^0)$, $\check{D}_3(\Sigma_1^0)$ and $\check{D}_2(\Sigma_2^0)$.

2 Finitely generated ω -powers.

Notation. In order to answer to Question (2), we set

$$\Sigma_0 := \{A \subseteq n^{<\omega} / A^\infty = \emptyset\}, \quad \Pi_0 := \{A \subseteq n^{<\omega} / A^\infty = n^\omega\},$$

$$\Delta_1 := \{A \subseteq n^{<\omega} / A^\infty \in \Delta_1^0\},$$

$$\Sigma_\xi := \{A \subseteq n^{<\omega} / A^\infty \in \Sigma_\xi^0\}, \quad \Pi_\xi := \{A \subseteq n^{<\omega} / A^\infty \in \Pi_\xi^0\} \quad (\xi \geq 1),$$

$$\Delta := \{A \subseteq n^{<\omega} / A^\infty \in \Delta_1^1\}.$$

- If $A \subseteq n^{<\omega}$, then we set $A^- := A \setminus \{\emptyset\}$.
- We define, for $s \in n^{<\omega}$ and $\alpha \in n^\omega$, $\alpha - s := (\alpha(|s|), \alpha(|s| + 1), \dots)$.
- If $\mathcal{S} \subseteq (n^{<\omega})^{<\omega}$, then we set $\mathcal{S}^* := \{S^* := S(0) \dots S(|s| - 1) / S \in \mathcal{S}\}$.

- We define a recursive map $\pi : n^\omega \times \omega^\omega \times \omega \rightarrow n^{<\omega}$ by

$$\pi(\alpha, \beta, q) := \begin{cases} (\alpha(0), \dots, \alpha(\beta[0])) & \text{if } q = 0, \\ (\alpha(1 + \sum_{j < q} \beta[j]), \dots, \alpha(\sum_{j \leq q} \beta[j])) & \text{otherwise.} \end{cases}$$

We always have the following equivalence:

$$\alpha \in A^\infty \Leftrightarrow \exists \beta \in \omega^\omega [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)].$$

Proposition 2 (*[S]*) $A^\infty \in \Sigma_1^1$ for all $A \subseteq n^{<\omega}$. If A is finite, then $A^\infty \in \Pi_1^0$.

Proof. We define a continuous map $c : (A^-)^\omega \rightarrow n^\omega$ by the formula $c((a_i)) := a_0 a_1 \dots$. We have $A^\infty = c[(A^-)^\omega]$, and $(A^-)^\omega$ is a Polish space (compact if A is finite). \square

Proposition 3 If $A^\infty \in \Delta_1^0$, then there exists a finite subset B of A such that $A^\infty = B^\infty$.

Proof. Set $E_k := \{\alpha \in n^\omega / \alpha[k \in A \text{ and } \alpha - \alpha[k \in A^\infty]\}$. It is an open subset of n^ω since A^∞ is open, and $A^\infty \subseteq \bigcup_{k > 0} E_k$. We can find an integer p such that $A^\infty \subseteq \bigcup_{0 < k \leq p} E_k$, by compactness of A^∞ . Let $B := A \cap n^{\leq p}$. If $\alpha \in A^\infty$, then we can find an integer $0 < k_0 \leq p$ such that $\alpha[k_0 \in A$ and $\alpha - \alpha[k_0 \in A^\infty$. Thus $\alpha[k_0 \in B$. Then we do it again with $\alpha - \alpha[k_0$, and so on. Thus we have $\alpha \in B^\infty = A^\infty$. \square

Remark. This is not true if we only assume that A^∞ is closed. Indeed, we have the following counterexample, due to O. Finkel:

$$A := \{s \in 2^{<\omega} / \forall i \leq |s| \ 2 \cdot \text{Card}(\{j < i / s(j) = 1\}) \geq i\}.$$

We have $A^\infty = \{\alpha \in 2^\omega / \forall i \in \omega \ 2 \cdot \text{Card}(\{j < i / \alpha(j) = 1\}) \geq i\}$ and if B is finite and $B^\infty = A^\infty$, $B \subseteq A$ and $101^2 0^2 \dots \notin B^\infty$.

Theorem 4 (a) $\Sigma_0 = \{\emptyset, \{\emptyset\}\}$ is Π_1^0 -complete.

(b) Π_0 is a dense Σ_1^0 subset of $2^{n^{<\omega}}$. In particular, Π_0 is Σ_1^0 -complete.

(c) Δ_1 is a $K_\sigma \setminus \Pi_2^0$ subset of $2^{n^{<\omega}}$. In particular, Δ_1 is Σ_2^0 -complete.

Proof. (a) Is clear.

(b) If we can find $m \in \omega$ with $n^m \subseteq A$, then $A^\infty = n^\omega$. As $\{A \subseteq n^{<\omega} / \exists m \in \omega \ n^m \subseteq A\}$ is a dense open subset of $2^{n^{<\omega}}$, the density follows. The formula

$$A \in \Pi_0 \Leftrightarrow \exists m \forall s \in n^m \exists q \leq m \ s[q \in A^-]$$

shows that Π_0 is Σ_1^0 , and comes from Proposition 3.

(c) If $A^\infty \in \Delta_1^0$, then we can find $p > 0$ such that $A^\infty = (A \cap n^{\leq p})^\infty$, by Proposition 3. So let $s_1, \dots, s_k, t_1, \dots, t_l \in 2^{<\omega}$ be such that $A^\infty = \bigcup_{1 \leq i \leq k} N_{s_i} = n^\omega \setminus (\bigcup_{1 \leq j \leq l} N_{t_j})$. For each $1 \leq j \leq l$, and for each sequence $s \in [(A^-)^{<\omega}]^* \setminus \{\emptyset\}$, $t_j \not\leq s$. So we have

$$A^\infty \in \Delta_1^0 \Leftrightarrow \begin{cases} \exists p > 0 \ \exists k, l \in \omega \ \exists s_1, \dots, s_k, t_1, \dots, t_l \in 2^{<\omega} \ \bigcup_{1 \leq i \leq k} N_{s_i} = n^\omega \setminus (\bigcup_{1 \leq j \leq l} N_{t_j}) \\ \text{and } \forall 1 \leq j \leq l \ \forall s \in [(A^-)^{<\omega}]^* \setminus \{\emptyset\} \ t_j \not\leq s \text{ and } \forall \alpha \in n^\omega \\ \{\alpha \notin \bigcup_{1 \leq i \leq k} N_{s_i} \text{ or } \exists \beta \in p^\omega [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)]\}. \end{cases}$$

This shows that Δ_1 is a K_σ subset of $2^{n^{<\omega}}$.

To show that it is not Π_2^0 , it is enough to see that its intersection with the closed set

$$\{A \subseteq n^{<\omega} / A^\infty \neq n^\omega\}$$

is dense and co-dense in this closed set (see (b)), by Baire's theorem. So let O be a basic clopen subset of $2^{n^{<\omega}}$ meeting this closed set. We may assume that it is of the form

$$\{A \subseteq n^{<\omega} / \forall i \leq k \ s_i \in A \text{ and } \forall j \leq l \ t_j \notin A\},$$

where $s_0, \dots, s_k, t_0, \dots, t_l \in n^{<\omega}$ and $|s_0| > 0$. Let $A := \{s_i / i \leq k\}$. Then $A \in O$ and A^∞ is in $\Pi_1^0 \setminus \{\emptyset, n^\omega\}$. There are two cases.

If $A^\infty \in \Delta_1^0$, then we have to find $B \in O$ with $B^\infty \notin \Delta_1^0$. Let $u_0, \dots, u_m \in n^{<\omega}$ with $\bigcup_{p \leq m} N_{u_p} = n^\omega \setminus A^\infty$. Let $r \in n \setminus \{u_0(|u_0| - 1)\}$, $s := u_0 r^{|u_0| + \max_{j \leq l} |t_j|}$ and $B := A \cup \{s\}$. Then $B \in O$ and $s^\infty \in B^\infty$. Let us show that s^∞ is not in the interior of B^∞ . Otherwise, we could find an integer q such that $N_{s^q} \subseteq B^\infty$. We would have $\alpha := s^q u_0 u_0(|u_0| - 1) r^\infty \in B^\infty$. As $N_{u_0} \cap A^\infty = \emptyset$, the decomposition of α into nonempty words of B would start with q times s . If this decomposition could go on, then we would have $u_0 = (u_0(|u_0| - 1))^{|u_0|}$. Let $v \in n^{<\omega}$ be such that $N_v \subseteq A^\infty$. We have $v(u_0(|u_0| - 1))^\infty \in A^\infty$, so $(u_0(|u_0| - 1))^\infty \in N_{u_0} \cap A^\infty$. But this is absurd. Therefore $B^\infty \notin \Delta_1^0$.

If $A^\infty \notin \Delta_1^0$, then we have to find $B \in O$ such that $B^\infty \in \Delta_1^0 \setminus \{n^\omega\}$. Notice that $n^\omega \neq \bigcup_{i \leq k} N_{s_i}$. So let $v \in n^{<\omega}$ be non constant such that $N_v \cap \bigcup_{i \leq k} N_{s_i} = \emptyset$. We set

$$D := A \cup \bigcup_{r \in n \setminus \{v(0)\}} \{(r)\} \cup \{v(0)^{|v|}\},$$

$B := A \cup \{s \in n^{<\omega} / |s| > \max_{j \leq l} |t_j| \text{ and } \exists t \in D \ t \prec s\}$. We get $B^\infty = \bigcup_{t \in D} N_t \in \Delta_1^0$ and

$$N_v \cap B^\infty = \emptyset,$$

so $B^\infty \neq n^\omega$. □

Now we will study $\mathcal{F} := \{A \subseteq n^{<\omega} / \exists B \subseteq n^{<\omega} \text{ finite } A^\infty = B^\infty\}$.

Proposition 5 \mathcal{F} is a co-nowhere dense Σ_2^0 -hard subset of $2^{n^{<\omega}}$.

Proof. By Proposition 3, if $A^\infty = n^\omega$, then there exists an integer p such that $A^\infty = (A \cap n^{\leq p})^\infty$, so $\Pi_0 \subseteq \mathcal{F}$ and, by Theorem 4, \mathcal{F} is co-nowhere dense. We define a continuous map $\phi : 2^\omega \rightarrow 2^{n^{<\omega}}$ by the formula $\phi(\gamma) := \{0^k 1 / \gamma(k) = 1\}$. If $\gamma \in P_f := \{\alpha \in 2^\omega / \exists p \ \forall m \geq p \ \alpha(m) = 0\}$, then $\phi(\gamma) \in \mathcal{F}$. If $\gamma \notin P_f$, then the concatenation map is an homeomorphism from $\phi(\gamma)^\omega$ onto $\phi(\gamma)^\infty$, thus $\phi(\gamma)^\infty$ is not K_σ . So $\phi(\gamma) \notin \mathcal{F}$, by Proposition 2. Thus the preimage of \mathcal{F} by ϕ is P_f , and \mathcal{F} is Σ_2^0 -hard. □

Let $\mathcal{G}_p := \{A \subseteq n^{<\omega} / \exists s_1, \dots, s_p \in n^{<\omega} \ A^\infty = \{s_1, \dots, s_p\}^\infty\}$, so that $\mathcal{F} = \bigcup_p \mathcal{G}_p$. We have $\mathcal{G}_0 = \Sigma_0$, so \mathcal{G}_0 is $\Pi_1^0 \setminus \Sigma_1^0$.

Proposition 6 \mathcal{G}_1 is $\Pi_1^0 \setminus \Sigma_1^0$. In particular, \mathcal{G}_1 is Π_1^0 -complete.

Proof. If $p \in \omega \setminus \{0\}$, then $\{0, 1^p\} \notin \mathcal{G}_1$ since $B^\infty = \{s^\infty\}$ if $B = \{s\}$. Thus $\{0\}$ is not an interior point of \mathcal{G}_1 since the sequence $(\{0, 1^p\})_{p>0}$ tends to $\{0\}$. So $\mathcal{G}_1 \notin \Sigma_1^0$.

- Let $(A_m) \subseteq \mathcal{G}_1$ tending to $A \subseteq n^{<\omega}$. If $A \subseteq \{\emptyset\}$, then $A^\infty = \emptyset = \{\emptyset\}^\infty$, so $A \in \mathcal{G}_1$. If $A \not\subseteq \{\emptyset\}$, then let $t \in A^-$ and $\alpha_0 := t^\infty$. There exists an integer m_0 such that $t \in A_m$ for $m \geq m_0$. Thus we may assume that $t \in A_m$ and $A_m^\infty \neq \emptyset$. So let $s_m \in n^{<\omega} \setminus \{\emptyset\}$ be such that $A_m^\infty = \{s_m\}^\infty = \{s_m^\infty\}$. We have $s_m^\infty = \alpha_0$. Let $b := \min\{a \in \omega \setminus \{0\} / (\alpha_0 \upharpoonright a)^\infty = \alpha_0\}$.

- We will show that $A_m \subseteq \{(\alpha_0 \upharpoonright b)^q / q \in \omega\}$. Let $s \in A_m \setminus \{\emptyset\}$. As $s^\infty = \alpha_0$, we can find an integer $a > 0$ such that $s = \alpha_0 \upharpoonright a$, and $b \leq a$. Let $r < b$ and q be integers so that $a = q \cdot b + r$. We have, if $r > 0$,

$$\begin{aligned} \alpha_0 &= (\alpha_0 \upharpoonright a)^\infty = (\alpha_0 \upharpoonright b)^\infty = (\alpha_0 \upharpoonright q \cdot b)^\infty = (\alpha_0 \upharpoonright a - \alpha_0 \upharpoonright q \cdot b)^\infty \alpha_0 \\ &= (\alpha_0 \upharpoonright b)^q (\alpha_0 \upharpoonright a - \alpha_0 \upharpoonright q \cdot b)^\infty \alpha_0 = (\alpha_0 \upharpoonright a - \alpha_0 \upharpoonright q \cdot b)^\infty \alpha_0 = (\alpha_0 \upharpoonright r)^\infty \alpha_0 = (\alpha_0 \upharpoonright r)^\infty. \end{aligned}$$

Thus, by minimality of b , $r = 0$ and we are done.

- Let $u \in A$. We can find an integer m_u such that $u \in A_m$ for $m \geq m_u$. So there exists an integer q_u such that $u = (\alpha_0 \upharpoonright b)^{q_u}$. Therefore $A^\infty = \{(\alpha_0 \upharpoonright b)^\infty\} = \{\alpha_0 \upharpoonright b\}^\infty$ and $A \in \mathcal{G}_1$. \square

Remark. Notice that this shows that we can find $w \in n^{<\omega} \setminus \{\emptyset\}$ such that $A \subseteq \{w^q / q \in \omega\}$ if $A \in \mathcal{G}_1$. Now we study \mathcal{G}_2 . The next lemma is just Corollary 6.2.5 in [Lo].

Lemma 7 Two finite sequences which commute are powers of the same finite sequence.

Proof. Let x and y be finite sequences with $xy = yx$. Then the subgroup of the free group on n generators generated by x and y is abelian, hence isomorphic to \mathbb{Z} . One generator of this subgroup must be a finite sequence u such that x and y are both powers of u . \square

Lemma 8 Let $A \in \mathcal{G}_2$. Then there exists a finite subset F of A such that $A^\infty = F^\infty$.

Proof. We will show more. Let $A \notin \mathcal{G}_1$ satisfying $A^\infty = \{s_1, s_2\}^\infty$, with $|s_1| \leq |s_2|$. Then

(a) The decomposition of α into words of $\{s_1, s_2\}$ is unique for each $\alpha \in A^\infty$ (this is a consequence of Corollaries 6.2.5 and 6.2.6 in [Lo]).

(b) $s_2 s_1 \perp s_1^q s_2$ for each integer $q > 0$, and $s_2 s_1 \wedge s_1^q s_2 = s_1 s_2 \wedge s_2 s_1$.

(c) $A \subseteq [\{s_1, s_2\}^{<\omega}]^*$.

- We prove the first two points. We split into cases.

2.1. $s_1 \perp s_2$.

The result is clear.

2.2. $s_1 \prec_{\neq} s_2 \not\prec s_1^\infty$.

Here also, the result is clear (cut α into words of length $|s_1|$).

2.3. $s_1 \prec_{\neq} s_2 \prec s_1^\infty$.

We can write $s_2 = s_1^m s$, where $m > 0$ and $s \prec_{\neq} s_1$. Thus $s_2 s_1 = s_1^m s s_1$ and $s_1^{m+1} s \prec s_1^q s_2$ if $q > 0$. But $s_1^m s s_1 \perp s_1^{m+1} s$ otherwise $s s_1 = s_1 s$, and $s, s_1 s_2$ would be powers of some sequence, which contradicts $A \notin \mathcal{G}_1$.

• We prove (c). Let $t \in A$, so that $ts_1^\infty, ts_2 s_1^\infty \in A^\infty$. These sequences split after $t(s_1 s_2 \wedge s_2 s_1)$, and the decomposition of ts_1^∞ (resp., $ts_2 s_1^\infty$) into words of $\{s_1, s_2\}$ starts with us_i (resp., us_{3-i}), where $u \in [\{s_1, s_2\}^{<\omega}]^*$. So ts_1^∞ and $ts_2 s_1^\infty$ split after $u(s_1 s_2 \wedge s_2 s_1)$ by (b). But we must have $t = u$ because of the position of the splitting point.

• We prove Lemma 8. If $A \in \mathcal{G}_0$, then $F := \emptyset$ works. If $A \in \mathcal{G}_1 \setminus \mathcal{G}_0$, then let $w \in n^{<\omega} \setminus \{\emptyset\}$ such that $A \subseteq \{w^q/q \in \omega\}$, and $q > 0$ such that $w^q \in A$. Then $F := \{w^q\}$ works. So we may assume that $A \notin \mathcal{G}_1$, and $A^\infty = \{s_1, s_2\}^\infty$. As $A^\infty \subseteq \bigcup_{t \in A^-} \{\alpha \in N_t/s_1 s_2 \wedge s_2 s_1 \prec \alpha - t\}$ is compact, we get a finite subset F of A^- such that $A^\infty \subseteq \bigcup_{t \in F} \{\alpha \in N_t/s_1 s_2 \wedge s_2 s_1 \prec \alpha - t\}$. We have $F^\infty \subseteq A^\infty$. If $\alpha \in A^\infty$, then let $t \in F$ such that $t \prec \alpha$. By (c), we have $t \in [\{s_1, s_2\}^{<\omega}]^*$. The sequence t is the beginning of the decomposition of α into words of $\{s_1, s_2\}$. Thus $\alpha - t \in A^\infty$ and we can go on like this. This shows that $\alpha \in F^\infty$. \square

Remark. The inclusion of $A^\infty = \{s_1, s_2\}^\infty$ into $\{t_1, t_2\}^\infty$ does not imply $\{s_1, s_2\} \subseteq [\{t_1, t_2\}^{<\omega}]^*$, even if $A \notin \mathcal{G}_1$. Indeed, take $s_1 := 01, s_2 := t_1 := 0$ and $t_2 := 10$. But we have

$$|t_1| + |t_2| \leq |s_1| + |s_2|,$$

which is the case in general:

Lemma 9 *Let $A, B \notin \mathcal{G}_1$ satisfying $A^\infty = \{s_1, s_2\}^\infty \subseteq B^\infty = \{t_1, t_2\}^\infty$. Then there is $j \in 2$ such that $|t_{1+i}| \leq |s_{1+[i+j \bmod 2]}|$ for each $i \in 2$. In particular, $|t_1| + |t_2| \leq |s_1| + |s_2|$.*

Proof. We may assume that $|s_1| \leq |s_2|$. Let, for $i = 1, 2$, $(w_m^i)_m \subseteq \{t_1, t_2\}$ be sequences such that $s_1^\infty = w_0^1 w_1^1 \dots$ (resp., $s_2 s_1^\infty = w_0^2 w_1^2 \dots$). By the proof of Lemma 8, there is a minimal integer m_0 satisfying $w_{m_0}^1 \neq w_{m_0}^2$. We let $u := w_0^1 \dots w_{m_0-1}^1$. The sequences s_1^∞ and $s_2 s_1^\infty$ split after $s_1 s_2 \wedge s_2 s_1 = u(t_1 t_2 \wedge t_2 t_1)$. Similarly, s_1^∞ and $s_1 s_2 s_1^\infty$ split after $s_1(s_1 s_2 \wedge s_2 s_1) = v(t_1 t_2 \wedge t_2 t_1)$, where $v \in [\{t_1, t_2\}^{<\omega}]^* \setminus \{\emptyset\}$. So we get $s_1 u = v$. Similarly, with the sequences $s_2 s_1^\infty$ and $s_2^2 s_1^\infty$, we see that $s_2 u \in [\{t_1, t_2\}^{<\omega}]^* \setminus \{\emptyset\}$. So we may assume that $u \neq \emptyset$ since $\{s_1, s_2\} \notin \mathcal{G}_1$. If $t_1 \not\prec t_2$, then we may assume that $\emptyset \neq t_1 \prec_{\neq} t_2$. So we may assume that we are not in the case $t_2 \prec t_1^\infty$. Indeed, otherwise $t_2 = t_1^m t$, where $\emptyset \prec_{\neq} t \prec_{\neq} t_1$ (see the proof of Lemma 8). Moreover, t_1 doesn't finish t_2 , otherwise we would have $t_1 = t(t_1 - t) = (t_1 - t)t$ and $t, t_1 - t, t_1, t_2$ would be powers of the same sequence, which contradicts $\{t_1, t_2\} \notin \mathcal{G}_1$. As $s_i u \in [\{t_1, t_2\}^{<\omega}]^*$, this shows that $s_i \in [\{t_1, t_2\}^{<\omega}]^*$. So we are done since $\{s_1, s_2\} \notin \mathcal{G}_1$ as before.

Assume for example that $t_2 = w_{m_0}^1$. Let m' be maximal with $t_1^{m'} \prec t_2$. Notice that

$$ut_1^{m'} \prec s_1 s_2 \prec s_1 s_2 s_1^\infty.$$

We have $ut_2 \prec s_1 s_2 s_1^\infty$, otherwise we would obtain $ut_1^{m'+1} \prec s_1 s_2 s_1^\infty \wedge s_2 s_1^\infty = s_1 s_2 \wedge s_2 s_1 \prec s_1^\infty$, which is absurd. So we get $|t_2| \leq |s_1|$ since $|u| + |t_2| + |t_1 t_2 \wedge t_2 t_1| \leq |s_1| + |s_1 s_2 \wedge s_2 s_1|$. Similarly, $|t_1| \leq |s_2|$ since $ut_1^{m'+1} \prec s_2^2 s_1^\infty$. The argument is similar if $t_2 = w_{m_0}^2$ (we get $|t_i| \leq |s_i|$ in this case for $i = 1, 2$). \square

Corollary 10 \mathcal{G}_2 is a $\check{D}_\omega(\Sigma_1^0) \setminus D_\omega(\Sigma_1^0)$ set. In particular, \mathcal{G}_2 is $\check{D}_\omega(\Sigma_1^0)$ -complete.

Proof. We will apply the Hausdorff derivation to $\mathcal{G} \subseteq 2^{n^{<\omega}}$. This means that we define a decreasing sequence $(F_\xi)_{\xi < \omega_1}$ of closed subsets of $2^{n^{<\omega}}$ as follows:

$$F_\xi := \overline{\left(\bigcap_{\eta < \xi} F_\eta \right)} \cap \mathcal{G} \text{ if } \xi \text{ is even, } \quad \overline{\left(\bigcap_{\eta < \xi} F_\eta \right)} \setminus \mathcal{G} \text{ if } \xi \text{ is odd.}$$

Recall that if ξ is even, then $F_\xi = \emptyset$ is equivalent to $\mathcal{G} \in D_\xi(\Sigma_1^0)$. Indeed, we set $U_\xi := \check{F}_\xi$. We have $U_{\xi+1} \setminus U_\xi = F_\xi \setminus F_{\xi+1} \subseteq \mathcal{G}$ if ξ is even and $U_{\xi+1} \setminus U_\xi \subseteq \check{\mathcal{G}}$ if ξ is odd. Similarly, $U_\xi \setminus (\bigcup_{\eta < \xi} U_\eta) \subseteq \check{\mathcal{G}}$ if ξ is limit. If $F_\xi = \emptyset$, then let η be minimal such that $F_\eta = \emptyset$. We have $\mathcal{G} = \bigcup_{\theta \leq \eta, \theta \text{ odd}} U_\theta \setminus (\bigcup_{\rho < \theta} U_\rho)$. If η is odd, then $\check{\mathcal{G}} = \bigcup_{\theta < \eta, \theta \text{ even}} U_\theta \setminus (\bigcup_{\rho < \theta} U_\rho) \in D_\eta(\Sigma_1^0)$, thus $\mathcal{G} \in \check{D}_\eta(\Sigma_1^0) \subseteq D_\xi(\Sigma_1^0)$. If η is even, then $\mathcal{G} = \bigcup_{\theta < \eta, \theta \text{ odd}} U_\theta \setminus (\bigcup_{\rho < \theta} U_\rho) \in D_\eta(\Sigma_1^0)$ and the same conclusion is true. Conversely, if $\mathcal{G} \in D_\xi(\Sigma_1^0)$, then let $(V_\eta)_{\eta < \xi}$ be an increasing sequence of open sets with $\mathcal{G} = \bigcup_{\eta < \xi, \eta \text{ odd}} V_\eta \setminus (\bigcup_{\theta < \eta} V_\theta)$. By induction, we check that $F_\eta \subseteq \check{V}_\eta$ if $\eta < \xi$. This clearly implies that $F_\xi = \emptyset$ because ξ is even.

• We will show that if $A \notin \mathcal{G}_1$ satisfies $A^\infty = \{s_1, s_2\}^\infty$, then $A \notin F_M := F_M(\mathcal{G}_2)$, where M is the smallest odd integer greater than or equal to $f(s_1, s_2) := 2\Sigma_{l \leq |s_1| + |s_2| - 2} n^{2(|s_1| + |s_2| - l)}$.

We argue by contradiction: A is the limit of (A_q) , where $A_q \in F_{M-1} \setminus \mathcal{G}_2$. Lemma 8 gives a finite subset F of A , and we may assume that $F \subseteq A_q$ for each q . Thus we have $A^\infty \subseteq A_q^\infty$, and the inclusion is strict. Thus we can find $s^q \in [A_q^{<\omega}]^*$ such that $N_{s^q} \cap A^\infty = \emptyset$. Let $s_0^q, \dots, s_{m_q}^q \in A_q$ be such that $s^q = s_0^q \dots s_{m_q}^q$.

Now A_q is the limit of $(A_{q,r})_r$, where $A_{q,r} \in F_{M-2} \cap \mathcal{G}_2$, and we may assume that

$$\{s_0^q, \dots, s_{m_q}^q\} \cup F \subseteq A_{q,r}$$

for each r , and that $A_{q,r} \notin \mathcal{G}_1$ because $A_q \notin \mathcal{G}_1 \subseteq \mathcal{G}_2$. Let $s_1^{q,r}, s_2^{q,r}$ such that $A_{q,r}^\infty = \{s_1^{q,r}, s_2^{q,r}\}^\infty$. By Lemma 9 we have $|s_1^{q,r}| + |s_2^{q,r}| \leq |s_1| + |s_2|$. Now we let $B_0 := A_{0,0}$ and $s_i^0 := s_i^{0,0}$ for $i = 1, 2$. We have $B_0 \in F_{M-2} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$, $A^\infty \not\subseteq B_0^\infty = \{s_1^0, s_2^0\}^\infty$, and

$$|s_1^0| + |s_2^0| \leq |s_1| + |s_2|.$$

Now we iterate this: for each $0 < k < n^{2(|s_1|+|s_2|)}$, we get $B_k \in F_{M-2(k+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$ such that $B_{k-1}^\infty \subsetneq B_k^\infty = \{s_1^k, s_2^k\}^\infty$ and $|s_1^k| + |s_2^k| \leq |s_1^{k-1}| + |s_2^{k-1}|$. We can find $k_0 < n^{2(|s_1|+|s_2|)}$ such that $|s_1^{k_0}| + |s_2^{k_0}| < |s_1^{k_0-1}| + |s_2^{k_0-1}|$ (with the convention $s_i^{-1} := s_i$). We set $C_0 := B_{k_0}$, $t_i^0 := s_i^{k_0}$. So we have $C_0 \in F_{M-2(k_0+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$, $C_0^\infty = \{t_1^0, t_2^0\}^\infty$ and $|t_1^0| + |t_2^0| < |s_1| + |s_2|$. Now we iterate this: for each $l \leq |s_1| + |s_2| - 2$, we get $t_1^l, t_2^l, k_l < n^{2(|t_1^{l-1}|+|t_2^{l-1}|)}$ and

$$C_l \in F_{M-2\Sigma_{m \leq l} (k_m+1)} \cap \mathcal{G}_2 \setminus \mathcal{G}_1$$

satisfying $C_l^\infty = \{t_1^l, t_2^l\}^\infty$ and $|t_1^l| + |t_2^l| < |t_1^{l-1}| + |t_2^{l-1}|$ (with the convention $t_i^{-1} := s_i$). We have $|t_1^l| + |t_2^l| \leq |s_1| + |s_2| - 1 - l$, thus

$$2\Sigma_{l \leq |s_1|+|s_2|-2} (k_l + 1) \leq 2\Sigma_{l \leq |s_1|+|s_2|-2} n^{2(|t_1^{l-1}|+|t_2^{l-1}|)} \leq f(s_1, s_2)$$

and this construction is possible. But we have $|t_1^{|s_1|+|s_2|-2}| + |t_2^{|s_1|+|s_2|-2}| \leq 1$, thus $C_{|s_1|+|s_2|-2} \in \mathcal{G}_1$, which is absurd.

Let $A \notin \mathcal{G}_2$. As $A \notin \mathcal{G}_1$, we can find $s, t \in A$ which are not powers of the same sequence. Indeed, let $s \in A^-$ and u with minimal length such that s is a power of u . Then any $t \in A \setminus \{u^q/q \in \omega\}$ works, because if s and t are powers of w , then w has to be a power of u . Indeed, as $u \prec w$, $w = u^k v$ with $v \prec u$, and v has to be a power of u by minimality of $|u|$ and Lemma 7. Assume that moreover $A \in F_{2k+2}$. Now A is the limit of $(A_{k,r})_r \subseteq F_{2k+1} \cap \mathcal{G}_2$ for each integer k , and we may assume that $s, t \in A_{k,r} \notin \mathcal{G}_1$. Let $s_1^{k,r}, s_2^{k,r}$ be such that $A_{k,r}^\infty = \{s_1^{k,r}, s_2^{k,r}\}^\infty$. By Lemma 9 we have $|s_1^{k,r}| + |s_2^{k,r}| \leq |s| + |t|$ and $f(s_1^{k,r}, s_2^{k,r}) \leq f(s, t)$. By the preceding point, we must have

$$2k + 1 < f(s, t).$$

Thus $\bigcap_m F_m \subseteq \mathcal{G}_2$. Notice that $F_{m+1}(\check{\mathcal{G}}_2) \subseteq F_m$, so that $F_\omega(\check{\mathcal{G}}_2) = \emptyset$ and $\mathcal{G}_2 \in \check{D}_\omega(\Sigma_1^0)$.

• Now let us show that $\{0\} \in F_\omega(\mathcal{G}_2)$ (this will imply $\mathcal{G}_2 \notin D_\omega(\Sigma_1^0)$). It is enough to see that

$$\{0\} \in \bigcap_m F_m.$$

Let $E(x)$ be the biggest integer less than or equal to x , $p_{k,s} := 2^{k+1-E(|s|/2)}$ and $k \in \omega$. We define $A_\emptyset := \{0\}$ and, for $s \in (\omega \setminus \{0, 1\})^{\leq 2k+1}$ and $m > 1$, $A_{sm} := A_s \cup \{(01^{p_{k,s}})^m; (0^2 1^{p_{k,s}})^m\}$ if $|s|$ is even, $A_s \cup \{s \in \{0, 1^{p_{k,s}}\}^{<\omega} / m \leq |s| \leq m + p_{k,s}\}$ if $|s|$ is odd. Let us show that $A_s \in \mathcal{G}_2$ (resp., $\check{\mathcal{G}}_2$) if $|s|$ is even (resp., odd). First by induction we get $A_{sm} \subseteq \{0, 1^{p_{k,s}}\}^{<\omega}$. Therefore $A_{sm}^\infty = \{0, 1^{p_{k,s}}\}^\infty$ if $|s|$ is odd, because if α is in $\{0, 1^{p_{k,s}}\}^\infty$ and $t \in \{0, 1^{p_{k,s}}\}^{<\omega}$ with minimal length $\geq m$ begins α , then $t \in A_{sm}$. Now if $|s|$ is even and $A_{sm}^\infty = \{s_1, s_2\}^\infty$, then $0^\infty \in \{s_1, s_2\}^\infty$, thus for example $s_1 = 0^{k+1}$. $(01^{p_{k,s}})^\infty \in \{s_1, s_2\}^\infty$, thus $s_2 \prec (01^{p_{k,s}})^\infty$ and $|s_2| \geq |(01^{p_{k,s}})^m|$ since $s_2 0^\infty \in \{s_1, s_2\}^\infty$. But then $(0^2 1^{p_{k,s}})^\infty \notin \{s_1, s_2\}^\infty$ since $m > 1$. Thus $A_{sm} \notin \mathcal{G}_2$.

As $(A_{sm})_m$ tends to A_s and $(A_s)_{|s|=2k+2} \subseteq \mathcal{G}_2$, we deduce from this that A_s is in $F_{2k+1-|s|} \setminus \mathcal{G}_2$ if $|s| \leq 2k + 1$ is odd, and that $A_s \in F_{2k+1-|s|} \cap \mathcal{G}_2$ if $|s| \leq 2k + 1$ is even. Therefore $\{0\}$ is in $\bigcap_k F_{2k+1} = \bigcap_m F_m$. \square

Remarks. (1) The end of this proof also shows that $\mathcal{G}_p \notin D_\omega(\Sigma_1^0)$ if $p \geq 2$. Indeed, $\{0\} \in F_\omega(\mathcal{G}_p)$. The only thing to change is the definition of A_{sm} if $|s|$ is even: we set

$$A_{sm} := A_s \cup \{(0^{j+1}1^{p_{k,s}})^m / j < p\}.$$

(2) If $\{s_1, s_2\} \notin \mathcal{G}_1$ and $\{s_1, s_2\}^\infty = \{t_1, t_2\}^\infty$, then $\{s_1, s_2\} = \{t_1, t_2\}$. Indeed, $\{t_1, t_2\} \notin \mathcal{G}_1$, thus by Lemma 9 we get $|s_1| + |s_2| = |t_1| + |t_2|$. By (c) in the proof of Lemma 8 and the previous fact, $s_i = t_{\varepsilon_i}^{a_i}$, where $a_i > 0$, ε_i , $i \in \{1, 2\}$. As $\{s_1, s_2\} \notin \mathcal{G}_1$, $\varepsilon_1 \neq \varepsilon_2$. Thus $a_i = 1$.

Conjecture 1. Let $A \in \mathcal{F}$. Then there exists a finite subset F of A such that $A^\infty = F^\infty$.

Conjecture 2. Let $p \geq 1$, $A, B \notin \mathcal{G}_p$ with $A^\infty = \{s_1, \dots, s_q\}^\infty \subseteq B^\infty = \{t_1, \dots, t_{p+1}\}^\infty$. Then $\Sigma_{1 \leq i \leq p+1} |t_i| \leq \Sigma_{1 \leq i \leq q} |s_i|$.

Conjecture 3. We have $\mathcal{G}_{p+1} \setminus \mathcal{G}_p \in D_\omega(\Sigma_1^0)$ for each $p \geq 1$. In particular, $\mathcal{F} \in K_\sigma \setminus \Pi_2^0$.

Notice that Conjectures 1 and 2 imply Conjecture 3. Indeed, $\mathcal{F} = \mathcal{G}_1 \cup \bigcup_{p \geq 1} \mathcal{G}_{p+1} \setminus \mathcal{G}_p$, so $\mathcal{F} \in K_\sigma$ if $\mathcal{G}_{p+1} \setminus \mathcal{G}_p \in D_\omega(\Sigma_1^0) \subseteq \Delta_2^0$, by Proposition 6. By Proposition 5 we have $\mathcal{F} \notin \Pi_2^0$. It is enough to see that $F_\omega := F_\omega(\mathcal{G}_{p+1} \setminus \mathcal{G}_p) = \emptyset$. We argue as in the proof of Corollary 10. This time, $f(s_1, \dots, s_q) := 2^{\Sigma_{l \leq \Sigma_{1 \leq i \leq q} |s_i| - 2}} n^{q(\Sigma_{1 \leq i \leq q} |s_i| - l)}$ for $s_1, \dots, s_q \in n^{<\omega}$. The fact to notice is that $A \notin F_M(\mathcal{G}_{p+1} \setminus \mathcal{G}_p)$ if $A \notin \mathcal{G}_p$ satisfies $A^\infty = \{s_1, \dots, s_{p+1}\}^\infty$ and M is the minimal odd integer greater than or equal to $f(s_1, \dots, s_{p+1})$. So if $A \in F_{2k+2} \cap \mathcal{F} \setminus \mathcal{G}_p$, then Conjecture 1 gives a finite subset $F := \{s_1, \dots, s_q\}$ of A . The set A is the limit of $(A_{k,r})_r \subseteq F_{2k+1} \cap \mathcal{G}_{p+1} \setminus \mathcal{G}_p$ for each integer k , and we may assume that $F \subseteq A_{k,r}$. Conjecture 2 implies that $f(s_1^{k,r}, \dots, s_{p+1}^{k,r}) \leq f(s_1, \dots, s_q)$ and $2k+1 < f(s_1, \dots, s_q)$. Thus $\bigcap_m F_m \subseteq \check{\mathcal{F}} \cup \mathcal{G}_p$. So $F_\omega \subseteq (\check{\mathcal{F}} \cup \mathcal{G}_p) \cap \mathcal{G}_{p+1} \setminus \mathcal{G}_p = \emptyset$.

3 Is A^∞ Borel?

Now we will see that the maximal complexity is possible. We essentially give O. Finkel's example, in a lightly simpler version.

Proposition 11 Let $\Gamma := \Sigma_1^1$ or a Baire class. The existence of $n \in \omega \setminus 2$ and $A \subseteq n^{<\omega}$ such that A^∞ is Γ -complete is equivalent to the existence of $B \subseteq 2^{<\omega}$ such that B^∞ is Γ -complete.

Proof. Let $p_n := \min\{p \in \omega / n \leq 2^p\} \geq 1$. We define $\phi : n \hookrightarrow 2^{p_n} := \{\sigma_0, \dots, \sigma_{2^{p_n}-1}\}$ by the formula $\phi(m) := \sigma_m$, $\Phi : n^{<\omega} \hookrightarrow 2^{<\omega}$ by the formula $\Phi(t) := \phi(t(0)) \dots \phi(t(|t| - 1))$ and $f : n^\omega \hookrightarrow 2^\omega$ by the formula $f(\gamma) := \phi(\gamma(0))\phi(\gamma(1)) \dots$. Then f is an homeomorphism from n^ω onto its range and reduces A^∞ to B^∞ , where $B := \Phi[A]$. The inverse function of f reduces B^∞ to A^∞ . So we are done if Γ is stable under intersection with closed sets. Otherwise, $\Gamma = \Delta_1^0$ or Σ_1^0 . If $A = \{s \in 2^{<\omega} / 0 \prec s \text{ or } 1^2 \prec s\}$, then $A^\infty = N_0 \cup N_{1^2}$, which is Δ_1^0 -complete. If $A = \{s \in 2^{<\omega} / 0 \prec s\} \cup \{10^k 1^{l+1} / k, l \in \omega\}$, then $A^\infty = 2^\omega \setminus \{10^\infty\}$, which is Σ_1^0 -complete. \square

Theorem 12 The set $I := \{(\alpha, A) \in n^\omega \times 2^{n^{<\omega}} / \alpha \in A^\infty\}$ is Σ_1^1 -complete. In fact,

- (a) (O. Finkel, see [F1]) There exists $A_0 \subseteq 2^{<\omega}$ such that A_0^∞ is Σ_1^1 -complete.
- (b) There exists $\alpha_0 \in 2^\omega$ such that I_{α_0} is Σ_1^1 -complete.

Proof. (a) We set $L := \{2, 3\}$ and

$$\mathcal{T} := \{ \tau \subseteq 2^{<\omega} \times L / \forall (u, \nu) \in 2^{<\omega} \times L [(u, \nu) \notin \tau] \text{ or}$$

$$[(\forall v \prec u \exists \mu \in L (v, \mu) \in \tau) \text{ and } ((u, 5 - \nu) \notin \tau) \text{ and } (\exists (\varepsilon, \pi) \in 2 \times L (u\varepsilon, \pi) \in \tau)] \}.$$

The set \mathcal{T} is the set of pruned trees over 2 with labels in L . It is a closed subset of $2^{2^{<\omega} \times L}$, thus a Polish space. Then we set

$$\sigma := \{ \tau \in \mathcal{T} / \exists (\underline{u}, \underline{\nu}) \in 2^\omega \times L^\omega [\forall m (\underline{u}[m], \underline{\nu}(m)) \in \tau] \text{ and } [\forall p \exists m \geq p \underline{\nu}(m) = 3] \}.$$

• Then $\sigma \in \Sigma_1^1(\mathcal{T})$. Let us show that it is complete. We set $\mathcal{T} := \{T \in 2^{\omega^{<\omega}} / T \text{ is a tree}\}$ and $IF := \{T \in \mathcal{T} / T \text{ is ill-founded}\}$. It is a well-known fact that \mathcal{T} is a Polish space (it is a closed subset of $2^{\omega^{<\omega}}$), and that IF is Σ_1^1 -complete (see [K1]). It is enough to find a Borel reduction of IF to σ (see [K2]).

We define $\psi : \omega^{<\omega} \hookrightarrow 2^{<\omega}$ by the formula $\psi(t) := 0^{t(0)} 10^{t(1)} 1 \dots 0^{t(|t|-1)} 1$, and $\Psi : \mathcal{T} \rightarrow \mathcal{T}$ by

$$\Psi(T) := \{ (u, \nu) \in 2^{<\omega} \times L / \exists t \in T u \prec \psi(t) \text{ and } \nu = 3 \text{ if } u = \emptyset, 2 + u(|u| - 1) \text{ otherwise} \}$$

$$\cup \{ (\psi(t)0^{k+1}, 2) / t \in T \text{ and } \forall q \in \omega tq \notin T, k \in \omega \}.$$

The map Ψ is Baire class one. Let us show that it is a reduction. If $T \in IF$, then let $\gamma \in \omega^\omega$ be such that $\gamma \upharpoonright m \in T$ for each integer m . We have $(\psi(\gamma \upharpoonright m), 3) \in \Psi(T)$. Let \underline{u} be the limit of $\psi(\gamma \upharpoonright m)$ and $\underline{\nu}(m) := 2 + \underline{u}(m - 1)$ (resp., 3) if $m > 0$ (resp., $m = 0$). These objects show that $\Psi(T) \in \sigma$. Conversely, we have $T \in IF$ if $\Psi(T) \in \sigma$.

• If $\tau \in \mathcal{T}$ and $m \in \omega$, then we enumerate $\tau \cap (2^m \times L) := \{(u_1^{m,\tau}, \nu_1^{m,\tau}), \dots, (u_{q_m,\tau}^{m,\tau}, \nu_{q_m,\tau}^{m,\tau})\}$ in the lexicographic ordering. We define $\varphi : \mathcal{T} \hookrightarrow 5^\omega$ by the formula

$$\varphi(\tau) := (u_1^{0,\tau} \nu_1^{0,\tau} \dots u_{q_0,\tau}^{0,\tau} \nu_{q_0,\tau}^{0,\tau} 4)(u_1^{1,\tau} \nu_1^{1,\tau} \dots u_{q_1,\tau}^{1,\tau} \nu_{q_1,\tau}^{1,\tau} 4) \dots$$

The set A_0 will be made of finite subsequences of sentences in $\varphi[\mathcal{T}]$. We set

$$A_0 := \{ u_{q+1}^{m,\tau} \nu_{q+1}^{m,\tau} \dots u_r^{p,\tau} \nu_r^{p,\tau} / \tau \in \mathcal{T}, m+1 < p, 0 \leq q \leq q_{m,\tau}, 1 \leq r \leq q_{p,\tau}, \\ [(m = 0 \text{ and } q = 0) \text{ or } (q > 0 \text{ and } \nu_q^{m,\tau} = 3 \text{ and } u_q^{m,\tau} \prec u_r^{p,\tau})], \nu_r^{p,\tau} = 3 \}$$

(with the convention $u_{q_{m,\tau}+1}^{m,\tau} \nu_{q_{m,\tau}+1}^{m,\tau} = 4$). It is clear that φ is continuous, and it is enough to see that it reduces σ to A_0^∞ .

So let us assume that $\tau \in \sigma$. This means the existence of an infinite branch in the tree with infinitely many 3 labels. We cut $\varphi(\tau)$ after the first 3 label of the branch corresponding to a sequence of length $m > 1$. Then we cut after the first 3 label corresponding to a sequence of length at least $m + 2$ of the branch. And so on. This clearly gives a decomposition of $\varphi(\tau)$ into words in A_0 .

If such a decomposition exists, then the first word is $u_1^{0,\tau} \nu_1^{0,\tau} \dots u_{r_0}^{p_0,\tau} \nu_{r_0}^{p_0,\tau}$, and the second is $u_{r_0+1}^{p_0,\tau} \nu_{r_0+1}^{p_0,\tau} \dots u_{r_1}^{p_1,\tau} \nu_{r_1}^{p_1,\tau}$. So we have $u_{r_0}^{p_0,\tau} \prec_{\neq} u_{r_1}^{p_1,\tau}$. And so on. This gives an infinite branch with infinitely many 3 labels.

• By Proposition 11, we can also have $A_0 \subseteq 2^{<\omega}$.

(b) Let $\alpha_0 := 1010^2 10^3 \dots$, (q_l) be the sequence of prime numbers: $q_0 := 2$, $q_1 := 3$, $M : \omega^{<\omega} \rightarrow \omega$ defined by $M_s := q_0^{s(0)+1} \dots q_{|s|-1}^{s(|s|-1)+1} + 1$, $\phi : \omega^{<\omega} \rightarrow 2^{<\omega} \setminus \{\emptyset\}$ defined by the formulas

$$\phi(\emptyset) := 1010^2 = 1010^{2M_\emptyset}$$

and $\phi(sm) := 10^{2M_s+1} 10^{2M_s+2} \dots 10^{2M_{sm}}$, and $\Phi : 2^{\omega^{<\omega}} \rightarrow 2^{n^{<\omega}}$ defined by $\Phi(T) := \phi[T]$.

• It is clear that $M_{sm} > M_s$, and that M and ϕ are well defined and one-to-one. So Φ is continuous:

$$\begin{aligned} s \in \phi[T] &\Leftrightarrow \exists t \ (t \in T \text{ and } \phi(t) = s) \\ &\Leftrightarrow s \in \phi[\omega^{<\omega}] \text{ and } \forall t \ (t \in T \text{ or } \phi(t) \neq s). \end{aligned}$$

If $T \in IF$, then we can find $\beta \in \omega^\omega$ such that $\phi(\beta \upharpoonright l) \in \Phi(T)$ for each integer l . Thus

$$\alpha_0 = (1010^{2M_{\beta \upharpoonright 0}})(10^{2M_{\beta \upharpoonright 0}+1} \dots 10^{2M_{\beta \upharpoonright 1}}) \dots \in (\Phi(T))^\infty.$$

Conversely, if $\alpha_0 \in (\Phi(T))^\infty$, then there exist $t_i \in T$ such that $\alpha_0 = \phi(t_0)\phi(t_1) \dots$. We have $t_0 = \emptyset$, and, if $i > 0$, then $M_{t_i \upharpoonright |t_i|-1} = M_{t_{i-1}}$; from this we deduce that $t_i \upharpoonright |t_i| - 1 = t_{i-1}$, because M is one-to-one. So let β be the limit of the t_i 's. We have $\beta \upharpoonright i = t_i$, thus $\beta \in [T]$ and $T \in IF$. Thus $\Phi \upharpoonright \mathcal{T}$ reduces IF to I_{α_0} . Therefore this last set is Σ_1^1 -complete. Indeed, it is clear that I is Σ_1^1 :

$$\alpha \in A^\infty \Leftrightarrow \exists \beta \in \omega^\omega \ [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)].$$

Finally, the map from \mathcal{T} into $n^\omega \times 2^{n^{<\omega}}$, which associates $(\alpha_0, \Phi(T))$ to T clearly reduces IF to I . So I is Σ_1^1 -complete. \square

Remark. This proof shows that if $\alpha = s_0 s_1 \dots$ and (s_i) is an antichain for the extension ordering, then I_α is Σ_1^1 -complete (here we have $s_i = 10^{2i+1} 10^{2i+2}$). To see it, it is enough to notice that $\phi(\emptyset) = s_0$ and $\phi(sm) = s_{M_s} \dots s_{M_{sm}-1}$. So I_α is Σ_1^1 -complete for a dense set of α 's.

We will deduce from this some true co-analytic sets. But we need a lemma, which has its own interest.

Lemma 13 (a) *The set A^∞ is Borel if and only if there exist a Borel function $f : n^\omega \rightarrow \omega^\omega$ such that*

$$\alpha \in A^\infty \Leftrightarrow (\forall m > 0 \ f(\alpha)(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, f(\alpha), q) \in A).$$

(b) *Let $\gamma \in \omega^\omega$ and $A \subseteq n^{<\omega}$. Then $A^\infty \in \Delta_1^1(A, \gamma)$ if and only if, for $\alpha \in n^\omega$, we have*

$$\alpha \in A^\infty \Leftrightarrow \exists \beta \in \Delta_1^1(A, \gamma, \alpha) \ [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)].$$

Proof. The “if” directions in (a) and (b) are clear. We have seen in the proof of Proposition 4 the “if” direction of the equivalences (the existence of an arbitrary β is necessary and sufficient). So let us show the “only if” directions.

(a) We define $f : \omega^\omega \rightarrow \omega^\omega$ by the formula $f(\alpha) := 0^\omega$ if $\alpha \notin A^\infty$, and, otherwise,

$$f(\alpha)(0) := \min\{p \in \omega / \alpha \upharpoonright (p+1) \in A \text{ and } \alpha - \alpha \upharpoonright (p+1) \in A^\infty\},$$

$$f(\alpha)(r+1) :=$$

$$\min\{k > 0 / [\alpha - \alpha \upharpoonright (1 + \sum_{j \leq r} f(\alpha)(j))] \upharpoonright k \in A \text{ and } \alpha - \alpha \upharpoonright (k+1 + \sum_{j \leq r} f(\alpha)(j)) \in A^\infty\}.$$

We get $\pi(\alpha, f(\alpha), 0) = \alpha \upharpoonright f(\alpha)(0) + 1 \in A$ and, if $q > 0$,

$$\pi(\alpha, f(\alpha), q) = (\alpha(1 + \sum_{j < q} f(\alpha)(j)), \dots, \alpha(\sum_{j \leq q} f(\alpha)(j))) \in A.$$

As f is clearly Borel, we are done.

(b) If $A^\infty \in \Delta_1^1(A, \gamma)$, then so is f and $\beta := f(\alpha) \in \Delta_1^1(A, \gamma, \alpha)$ is what we were looking for. \square

Remark. Lemma 13 is a particular case of a more general situation. Actually we have the following uniformization result. It was written after a conversation with G. Debs.

Proposition 14 *Let X and Y be Polish spaces, and $F \in \Pi_2^0(X \times Y)$ such that the projection $\Pi_X[F \cap (X \times V)]$ is Borel for each $V \in \Sigma_1^0(Y)$. Then there exists a Borel map $f : X \rightarrow Y$ such that $(x, f(x)) \in F$ for each $x \in \Pi_X[F]$.*

Proof. Let (Y_n) be a basis for the topology of Y with $Y_0 := Y$, $B_n := \Pi_X[F \cap (X \times Y_n)]$, and τ be a finer 0-dimensional Polish topology on X making the B_n 's clopen (see 13.5 in [K1]). We equip X with a complete τ -compatible metric d . Let $(O_m) \subseteq \Sigma_1^0(X \times Y)$ be decreasing satisfying $O_0 := X \times Y$ and $F = \bigcap_m O_m$. We construct a sequence $(U_s)_{s \in \omega^{<\omega}}$ of clopen subsets of $[B_0, \tau]$ with $U_\emptyset := B_0$, and a sequence $(V_s)_{s \in \omega^{<\omega}}$ of basic open sets of Y satisfying

- (a) $U_s \subseteq \Pi_X[F \cap (U_s \times V_s)]$
- (b) $\text{diam}_d(U_s), \text{diam}(V_s) \leq \frac{1}{|s|}$ if $s \neq \emptyset$
- (c) $U_s = \bigcup_{m, \text{disj.}} U_{s \frown m}, \overline{V_{s \frown n}} \subseteq V_s$
- (d) $U_s \times V_s \subseteq O_{|s|}$

• Assume that this construction has been achieved. If $x \notin B_0$, then we set $f(x) := y_0 \in Y$ (we may assume that $F \neq \emptyset$). Otherwise, we can find a unique sequence $\gamma \in \omega^\omega$ such that $x \in U_{\gamma \upharpoonright m}$ for each integer m . Thus we can find $y \in V_{\gamma \upharpoonright m}$ such that $(x, y) \in F$, and $(\overline{V_{\gamma \upharpoonright m}})_m$ is a decreasing sequence of nonempty closed sets whose diameters tend to 0, which defines a continuous map $f : [B_0, \tau] \rightarrow Y$. If $x \in B_0$, then $(x, f(x)) \in U_{\gamma \upharpoonright m} \times V_{\gamma \upharpoonright m} \subseteq O_m$, thus $\text{Gr}(f|_{B_0}) \subseteq F$. Notice that $f : [X, \tau] \rightarrow Y$ is continuous, so $f : X \rightarrow Y$ is Borel.

• Let us show that the construction is possible. We set $U_\emptyset := B_0$ and $V_\emptyset := Y$. Assume that $(U_s)_{s \in \omega^{\leq p}}$ and $(V_s)_{s \in \omega^{\leq p}}$ satisfying conditions (a)-(d) have been constructed, which is the case for $p = 0$. Let $s \in \omega^p$. If $(x, y) \in F \cap (U_s \times V_s)$, then we can find $U_x \in \Delta_1^0(U_s)$ and a basic open set $V_y \subseteq Y$ such that $(x, y) \in U_x \times V_y \subseteq U_s \times \overline{V_y} \subseteq (U_s \times V_s) \cap O_{p+1}$, and whose diameters are at most $\frac{1}{p+1}$. By the Lindelöf property, we can write $F \cap (U_s \times V_s) \subseteq \bigcup_n U_{x_n} \times V_{y_n}$ and $F \cap (U_s \times V_s) = \bigcup_n F \cap (U_{x_n} \times V_{y_n})$.

If $x \in U_s$, then let n and y be such that $(x, y) \in F \cap (U_{x_n} \times V_{y_n})$. Then

$$x \in O^n := \Pi_X[F \cap (X \times V_{y_n})] \cap U_{x_n} \in \Delta_1^0([B_0, \tau]).$$

Thus $U_s = \bigcup_n O^n$. We set $U_{s \smallfrown n} := O^n \setminus (\bigcup_{p < n} O^p)$ and $V_{s \smallfrown n} := V_{y_n}$, and we are done. \square

In our context, $F = \{(\alpha, \beta) \in n^\omega \times \omega^\omega / (\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)\}$, which is a closed subset of $X \times Y$. The projection $\Pi_X[F \cap (X \times N_s)]$ is Borel if A^∞ is Borel, since it is $\{S^* \gamma / S \in (A \cap n^{s(0)+1}) \times \Pi_{0 < j < |s|} (A \cap n^{s(j)}) \text{ and } \gamma \in A^\infty\}$.

Theorem 15 *The following sets are $\Pi_1^1 \setminus \Delta_1^1$:*

- (a) $\Pi := \{(A, \gamma, \theta) \in 2^{n^{<\omega}} \times \omega^\omega \times \omega^\omega / \theta \in \text{WO and } A^\infty \in \Pi_{|\theta|}^0 \cap \Delta_1^1(A, \gamma)\}$. The same thing is true with $\Sigma := \{(A, \gamma, \theta) \in 2^{n^{<\omega}} \times \omega^\omega \times \omega^\omega / \theta \in \text{WO and } A^\infty \in \Sigma_{|\theta|}^0 \cap \Delta_1^1(A, \gamma)\}$.
- (b) $\Sigma_1 := \{A \in 2^{n^{<\omega}} / A^\infty \in \Sigma_1^0 \cap \Delta_1^1(A)\}$. In fact, $\Sigma_\xi := \{A \in 2^{n^{<\omega}} / A^\infty \in \Sigma_\xi^0 \cap \Delta_1^1(A)\}$ is $\Pi_1^1 \setminus \Delta_1^1$ if $1 \leq \xi < \omega_1$. Similarly, $\Pi_\xi := \{A \in 2^{n^{<\omega}} / A^\infty \in \Pi_\xi^0 \cap \Delta_1^1(A)\}$ is $\Pi_1^1 \setminus \Delta_1^1$ if $2 \leq \xi < \omega_1$.
- (c) $\Delta := \{A \in 2^{n^{<\omega}} / A^\infty \in \Delta_1^1(A)\}$.

Proof. Consider the way of coding the Borel sets used in [Lou]. By Lemma 13 we get

$$(A, \gamma, \theta) \in \Pi \Leftrightarrow \begin{cases} \exists p \in \omega \ P(p, A, \gamma, \theta) \text{ and } \forall \alpha \in n^\omega \\ (\alpha \notin A^\infty \text{ or } (p, A, \gamma, \alpha) \in C) \text{ and } [(p, A, \gamma) \in W \text{ and } (p, A, \gamma, \alpha) \notin C] \text{ or} \\ \exists \beta \in \Delta_1^1(A, \gamma, \alpha) \ [(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)] \end{cases}$$

This shows that Π is Π_1^1 . The same argument works with Σ . From this we can deduce that Σ_1 is Π_1^1 , if we forget γ and take the section of Σ at $\theta \in \text{WO} \cap \Delta_1^1$ such that $|\theta| = 1$. Similarly, Σ_ξ and Π_ξ are co-analytic if $\xi \geq 1$. Forgetting θ , we see that the relation “ $A^\infty \in \Delta_1^1(A, \gamma)$ ” is Π_1^1 .

• Let us look at the proof of Theorem 12. We will show that if $\xi \geq 1$ (resp., $\xi \geq 2$), then $\Sigma_\xi \setminus I_{\alpha_0}$ (resp., $\Pi_\xi \setminus I_{\alpha_0}$) is a true co-analytic set. To do this, we will reduce WF to $\Sigma_\xi \setminus I_{\alpha_0}$ (resp., $\Pi_\xi \setminus I_{\alpha_0}$) in a Borel way. We change the definition of Φ . We set

$$t \subseteq \alpha_0 \Leftrightarrow \exists k \ t \prec \alpha_0 - \alpha_0 \upharpoonright k,$$

$$E := \{(\alpha_0 \upharpoonright p)r/p \in \omega \setminus \{2\}, r \in n \setminus \{\alpha_0(p)\}\}, \quad F := \{U^* \not\subseteq \alpha_0 / U \in \phi[T]^{<\omega}\},$$

$$\Phi'(T) := \phi[T] \cup \{s \in n^{<\omega} / \exists t \in E \cup F \ t \prec s\}.$$

This time, Φ' is Baire class one, since

$$s \in \Phi'(T) \Leftrightarrow s \in \phi[T] \text{ or } \exists t \in E \ t \prec s \text{ or } \exists U \in (2^{<\omega})^{<\omega} \ (\forall j < |u| \ U(j) \in \phi[T]) \text{ and } U^* \not\subseteq \alpha_0 \text{ and } U^* \prec s.$$

The proof of Theorem 12 remains valid, since if $\alpha_0 \in (\Phi'(T))^\infty$, then the decompositions of α_0 into words of $\Phi'(T)$ are actually decompositions into words of $\phi[T]$.

- Let us show that $(\Phi'(T))^\infty \in \Sigma_1^0 \cap \Delta_1^1(\Phi'(T))$ if $T \in WF$. The set $(\Phi'(T))^\infty$ is

$$\bigcup_{S \in \phi[T]^{<\omega}, l \in n \setminus \{1\}, m \in n \setminus \{0\}} [(\bigcup_{s/\exists t \in F t \prec s} N_{S^*s}) \cup N_{S^*l} \cup N_{S^*1m} \cup (N_{S^*101} \setminus \{S^*\alpha_0\})].$$

If $\alpha \in n^\omega$, then α contains infinitely many $l \in n \setminus \{1\}$ or finishes with 1^∞ . As 1^2 and the sequences beginning with l are in $\Phi'(T)$, the clopen sets are subsets of $(\Phi'(T))^\infty$ since $\phi[T]$ and the sequences beginning with $t \in F, l$ or $1m$ are in $\Phi'(T)$. If $\alpha \in N_{S^*101} \setminus \{S^*\alpha_0\}$, then let $p \geq 3$ be maximal such that $\alpha \upharpoonright (|S^*| + p) = S^*(\alpha_0 \upharpoonright p)$. We have $\alpha \in (\Phi'(T))^\infty$ since the sequences beginning with $(\alpha_0 \upharpoonright p)r$ are in $\Phi'(T)$. Thus we get the inclusion into $(\Phi'(T))^\infty$.

If $\alpha \in (\Phi'(T))^\infty$, then $\alpha = a_0 a_1 \dots$, where $a_i \in \Phi'(T)$. Either for all i we have $a_i \in \phi[T]$. In this case, there is i such that $a_0 \dots a_i \not\leq \alpha_0$, otherwise we could find k with $\alpha_0 - \alpha_0 \upharpoonright k \in (\Phi(T))^\infty$. But this contradicts the fact that $T \in WF$, as in the proof of Theorem 12. So we have $\alpha \in \bigcup_{\exists t \in F t \prec s} N_s$. Or there exists i minimal such that $a_i \notin \phi[T]$. In this case,

- Either $\exists t \in E t \prec a_i$ and $\alpha \in \bigcup_{S \in \phi[T]^{<\omega}, l \in n \setminus \{1\}, m \in n \setminus \{0\}} [N_{S^*l} \cup N_{S^*1m} \cup (N_{S^*101} \setminus \{S^*\alpha_0\})]$,
- Or $\exists t \in F t \prec a_i$ and $\alpha \in \bigcup_{S \in \phi[T]^{<\omega}} \bigcup_{s/\exists t \in F t \prec s} N_{S^*s}$.

From this we deduce that $(\Phi'(T))^\infty$ is Σ_1^0 .

Finally, we have

$$\alpha \in (\Phi'(T))^\infty \Leftrightarrow \begin{cases} \exists t \in n^{<\omega} \exists b \in \omega^{<\omega} [(|t| = 1 + \sum_{j < |b|} b(j)) \text{ and } (\forall 0 < m < |b| b(m) > 0) \\ \text{and } (\forall q < |b| \pi(t0^\infty, b0^\infty, q) \in \Phi'(T))] \text{ and } [\exists l \in n \setminus \{1\} tl \prec \alpha \text{ or } t1^2 \prec \alpha]. \end{cases}$$

This shows that $(\Phi'(T))^\infty$ is $\Delta_1^1(\Phi'(T))$.

Therefore, $\Phi'_{\upharpoonright T}$ reduces WF to $\Sigma_\xi \setminus I_{\alpha_0}$ if $\xi \geq 1$, and to $\Pi_\xi \setminus I_{\alpha_0}$ if $\xi \geq 2$. So these sets are true co-analytic sets. But $\Sigma_1 \cap I_{\alpha_0}$ is Π_1^1 , by Lemma 13. As $\Sigma_1 \setminus I_{\alpha_0} = \Sigma_1 \setminus (\Sigma_1 \cap I_{\alpha_0})$, Σ_1 is not Borel. Thus Σ is not Borel, as before. The argument is similar for Σ_ξ, Π_ξ ($\xi \geq 2$) and Π . And for Δ too. \square

Question. Does $A^\infty \in \Delta_1^1$ imply $A^\infty \in \Delta_1^1(A)$? Probably not. If the answer is positive, Δ , and more generally Σ_ξ (for $\xi \geq 1$) and Π_ξ (for $\xi \geq 2$) are true co-analytic sets.

Remark. In any case, Δ is Σ_2^1 because “ $A^\infty \in \Delta_1^1$ ” is equivalent to “ $\exists \gamma \in \omega^\omega A^\infty \in \Delta_1^1(A, \gamma)$ ”. This argument shows that Σ_ξ and Π_ξ are $\Sigma_2^1(\theta)$, where $\theta \in WO$ satisfies $|\theta| = \xi$. We can say more about Π_1 : it is Δ_2^1 . Indeed, in [St2] we have the following characterization:

$$A^\infty \in \Pi_1^0 \Leftrightarrow \forall \alpha \in n^\omega [\forall s \in n^{<\omega} (s \prec \alpha \Rightarrow \exists S \in A^{<\omega} s \prec S^*)] \Rightarrow \alpha \in A^\infty.$$

This gives a Π_2^1 definition of Π_1 . The same fact is true for Σ_1 :

Proposition 16 Σ_1 and Π_1 are co-nowhere dense $\Delta_2^1 \setminus D_2(\Sigma_1^0)$ subsets of $2^{n^{<\omega}}$. If $\xi \geq 2$, then Σ_ξ and Π_ξ are co-nowhere dense $\Sigma_2^1 \setminus D_2(\Sigma_1^0)$ subsets of $2^{n^{<\omega}}$. Δ is a co-nowhere dense $\Sigma_2^1 \setminus D_2(\Sigma_1^0)$ subset of $2^{n^{<\omega}}$.

Proof. We have seen that Σ_1 is Σ_2^1 ; it is also Π_2^1 because

$$A^\infty \in \Sigma_1^0 \Leftrightarrow \forall \alpha \in n^\omega \quad \alpha \notin A^\infty \text{ or } \exists s \in n^{<\omega} [s \prec \alpha \text{ and } \forall \beta \in n^\omega (s \not\prec \beta \text{ or } \beta \in A^\infty)].$$

By Proposition 4, Π_0 is co-nowhere dense, and it is a subset of $\Sigma_\xi \cap \Pi_\xi \cap \Delta$. So Σ_ξ , Π_ξ and Δ are co-nowhere dense, and it remains to see that they are not open. It is enough to notice that \emptyset is not in their interior. Look at the proof of Theorem 12; it shows that for each integer m , there is a subset A_m of $\{s \in 5^{<\omega} / |s| \geq m\}$ such that $A_m^\infty \notin \Delta_1^1$. But the argument in the proof of Proposition 11 shows that we can have the same thing in $n^{<\omega}$ for each $n \geq 2$. This gives the result because the sequence (A_m) tends to \emptyset . \square

We can say a bit more about Π_1 and Σ_2 :

Proposition 17 Π_1 , Π_1 and Σ_2 are Σ_2^0 -hard (so they are not Π_2^0).

Proof. Consider the map ϕ defined in the proof of Proposition 5. By Proposition 2, if $\gamma \in P_f$, then $\phi(\gamma)^\infty$ is Π_1^0 . Moreover, as $\phi(\gamma)$ is an antichain for the extension ordering, the decomposition into words of $\phi(\gamma)$ is unique. This shows that $\phi(\gamma)^\infty$ is Δ_1^1 , because

$$\alpha \in \phi(\gamma)^\infty \Leftrightarrow \exists \beta \in \Delta_1^1(\alpha) [(\forall m > 0 \quad \beta(m) > 0) \text{ and } (\forall q \in \omega \quad \pi(\alpha, \beta, q) \in \phi(\gamma))].$$

So $\phi(\gamma) \in \Pi_1$ if $\gamma \in P_f$. So the preimage of any of the sets in the statement by ϕ is P_f , and the result follows. \square

4 Which sets are ω -powers?

Now we come to Question (3). Let us specify what we mean by “codes for Γ -sets”, where Γ is a given class, and fix some notation.

- For the Borel classes, we will essentially consider the 2^ω -universal sets used in [K1] (see Theorem 22.3). For $\xi \geq 1$, $\mathcal{U}^{\xi, \mathcal{A}}$ (resp. $\mathcal{U}^{\xi, \mathcal{M}}$) is 2^ω -universal for $\Sigma_\xi^0(n^\omega)$ (resp. $\Pi_\xi^0(n^\omega)$). So we have

- $\mathcal{U}^{1, \mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega \quad \gamma(p) = 0 \text{ and } s_p^n \prec \alpha\}$, where $(s_p^n)_p$ enumerates $n^{<\omega}$.
- $\mathcal{U}^{\xi, \mathcal{M}} = \neg \mathcal{U}^{\xi, \mathcal{A}}$, for each $\xi \geq 1$.
- $\mathcal{U}^{\xi, \mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega \quad ((\gamma)_p, \alpha) \in \mathcal{U}^{\eta, \mathcal{M}}\}$ if $\xi = \eta + 1$.
- $\mathcal{U}^{\xi, \mathcal{A}} = \{(\gamma, \alpha) \in 2^\omega \times n^\omega / \exists p \in \omega \quad ((\gamma)_p, \alpha) \in \mathcal{U}^{\eta_p, \mathcal{M}}\}$ if ξ is the limit of the strictly increasing sequence of odd ordinals (η_p) .

- For the class Σ_1^1 , we fix some bijection $p \mapsto ((p)_0, (p)_1)$ between ω and ω^2 . We set

$$(\gamma, \alpha) \in \mathcal{U} \Leftrightarrow \exists \beta \in 2^\omega \quad (\forall m \exists p \geq m \quad \beta(p) = 1) \text{ and } (\forall p \quad [\gamma(p) = 1 \text{ or } s_{(p)_0}^2 \not\prec \beta \text{ or } s_{(p)_1}^n \not\prec \alpha]).$$

It is not hard to see that \mathcal{U} is 2^ω -universal for $\Sigma_1^1(n^\omega)$, and we use it here because of the compactness of $2^\omega \times n^\omega$, rather than the ω^ω -universal set for $\Sigma_1^1(n^\omega)$ given in [K1] (see Theorem 14.2).

- For the class Δ_1^1 , it is different because there is no universal set. But we can use the Π_1^1 set of codes $D \subseteq 2^\omega$ for the Borel sets in [K1] (see Theorem 35.5). We may assume that D , S and P are effective, by [M].

- The sets we are interested in are the following:

$$\mathcal{A}_\xi := \{\gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathcal{A}} \text{ is an } \omega\text{-power}\}, \quad \mathcal{M}_\xi := \{\gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathcal{M}} \text{ is an } \omega\text{-power}\}$$

$$\mathcal{B} := \{d \in D / D_d \text{ is an } \omega\text{-power}\},$$

$$\mathcal{A} := \{\gamma \in 2^\omega / \mathcal{U}_\gamma \text{ is an } \omega\text{-power}\}.$$

As we mentioned in the introduction, Lemma 13 is also related to Question (3). A rough answer to this question is Σ_3^1 . Indeed, we have, for $\gamma \in 2^\omega$,

$$\gamma \in \mathcal{A} \Leftrightarrow \exists A \in 2^{n^{<\omega}} \forall \alpha \in n^\omega ([(\gamma, \alpha) \notin \mathcal{U} \text{ or } \alpha \in A^\infty] \text{ and } [\alpha \notin A^\infty \text{ or } (\gamma, \alpha) \in \mathcal{U}]).$$

With Lemma 13, we have a better estimation of the complexity of \mathcal{B} : it is Σ_2^1 . Indeed, for $d \in D$,

$$D_d \text{ is an } \omega\text{-power} \Leftrightarrow \exists A \in 2^{n^{<\omega}} \forall \alpha \in n^\omega ([(d, \alpha) \notin S \text{ or } \exists \beta \in \Delta_1^1(A, d, \alpha)$$

$$[(\forall m > 0 \ \beta(m) > 0) \text{ and } (\forall q \in \omega \ \pi(\alpha, \beta, q) \in A)]) \text{ and } [\alpha \notin A^\infty \text{ or } (d, \alpha) \in P]).$$

This argument also shows that \mathcal{A}_ξ and \mathcal{M}_ξ are Σ_2^1 . We can say more about these two sets.

Proposition 18 *If $1 \leq \xi < \omega_1$, then \mathcal{A}_ξ and \mathcal{M}_ξ are $\Sigma_2^1 \setminus D_2(\Sigma_1^0)$ co-meager subsets of 2^ω . If moreover $\xi = 1$, then they are co-nowhere dense.*

Proof. We set $E_1 := \{\gamma \in 2^\omega / \mathcal{U}_\gamma^{1, \mathcal{A}} = n^\omega\}$, $E_{\eta+1} := \{\gamma \in 2^\omega / \forall p (\gamma)_p \in E_\eta\}$ if $\eta \geq 1$, and $E_\xi := \{\gamma \in 2^\omega / \forall p (\gamma)_p \in E_{\eta_p}\}$ (where (η_p) is a strictly increasing sequence of odd ordinals cofinal in the limit ordinal ξ). If $s \in 2^{<\omega}$, then we set $\gamma(p) = s(p)$ if $p < |s|$, 0 otherwise. Then $s \prec \gamma$ and $\mathcal{U}_\gamma^{1, \mathcal{A}} = n^\omega$, so E_1 is dense. If $\gamma_0 \in E_1$, then for all $\alpha \in n^\omega$ we can find an integer p such that $\gamma_0(p) = 0$ and $s_p^n \prec \alpha$. By compactness of n^ω we can find a finite subset F of $\{p \in \omega / \gamma_0(p) = 0\}$ such that for each $\alpha \in n^\omega$, $s_p^n \prec \alpha$ for some $p \in F$. Now $\{\gamma \in 2^\omega / \forall p \in F \ \gamma(p) = 0\}$ is an open neighborhood of γ_0 and a subset of E_1 . So E_1 is an open subset of 2^ω . Now the map $\gamma \mapsto (\gamma)_p$ is continuous and open, so $E_{\eta+1}$ and E_ξ are dense G_δ subsets of 2^ω . Then we notice that E_ξ is a subset of $\{\gamma \in 2^\omega / \mathcal{U}_\gamma^{\xi, \mathcal{A}} = n^\omega\}$ (resp., $\{\gamma \in 2^\omega / \mathcal{U}_\gamma^{1, \mathcal{A}} = \emptyset\}$) if ξ is odd (resp., even). Indeed, this is clear for $\xi = 1$. Then we use the formulas $\mathcal{U}_\gamma^{\eta+1, \mathcal{A}} = \bigcup_p \neg \mathcal{U}_{(\gamma)_p}^{\eta, \mathcal{A}}$ and $\mathcal{U}_\gamma^{\xi, \mathcal{A}} = \bigcup_p \neg \mathcal{U}_{(\gamma)_p}^{\eta_p, \mathcal{A}}$, and by induction we are done. As \emptyset and n^ω are ω -powers, we get the results about Baire category. Now it remains to see that \mathcal{A}_ξ and \mathcal{M}_ξ are not open. But by induction again $1^\infty \in \mathcal{A}_\xi \cap \mathcal{M}_\xi$, so it is enough to see that 1^∞ is not in the interior of these sets.

- Let us show that, for $O \in \Delta_1^0(n^\omega) \setminus \{\emptyset, n^\omega\}$ and for each integer m , we can find $\gamma, \gamma' \in 2^\omega$ such that $\gamma(j) = \gamma'(j) = 1$ for $j < m$, $\mathcal{U}_\gamma^{\xi, \mathcal{A}} = O$ and $\mathcal{U}_{\gamma'}^{\xi, \mathcal{M}} = O$.

For $\xi = 1$, write $O = \bigcup_p N_{s_{q_k}^n}$, where $q_k \geq m$. Let $\gamma(q) := 0$ if there exists k such that $q = q_k$, $\gamma(q) := 1$ otherwise. The same argument applied to \tilde{O} gives the complete result for $\xi = 1$.

Now we argue by induction. Let $\gamma_p \in 2^\omega$ be such that $\gamma_p(q) = 1$ for $\langle p, q \rangle < m$ and $\mathcal{U}_{(\gamma)_p}^{\eta, \mathcal{M}} = O$. Then define γ by $\gamma(\langle p, q \rangle) := \gamma_p(q)$; we have $\gamma(j) = 1$ if $j < m$ and $\mathcal{U}_\gamma^{\eta+1, \mathcal{A}} = \bigcup_p \mathcal{U}_{(\gamma)_p}^{\eta, \mathcal{M}} = O$. The argument with \check{O} still works. The argument is similar for limit ordinals.

• Now we apply this fact to $O := N_{(0)}$. This gives $\gamma_p, \gamma'_p \in N_{1^p}$ such that $\mathcal{U}_{\gamma_p}^{\xi, \mathcal{A}} = N_{(0)}$ and $\mathcal{U}_{\gamma'_p}^{\xi, \mathcal{M}} = N_{(0)}$. But $(\gamma_p), (\gamma'_p)$ tend to 1^∞ , $\gamma_p \notin \mathcal{A}_\xi$ and $\gamma'_p \notin \mathcal{M}_\xi$. \square

Corollary 19 \mathcal{A}_1 is $\check{D}_2(\Sigma_1^0) \setminus D_2(\Sigma_1^0)$. In particular, \mathcal{A}_1 is $\check{D}_2(\Sigma_1^0)$ -complete.

Proof. By the preceding proof, it is enough to see that $\mathcal{A}_1 \setminus \{1^\infty\}$ is open. So let $\gamma_0 \in \mathcal{A}_1 \setminus \{1^\infty\}$, p_0 in ω with $\gamma_0(p_0) = 0$, and $A_0 \subseteq n^{<\omega}$ with $\mathcal{U}_{\gamma_0}^{1, \mathcal{A}} = A_0^\infty$. If $\alpha \in n^\omega$, then $s_{p_0}^n \alpha \in \mathcal{U}_{\gamma_0}^{1, \mathcal{A}}$, so we can find $m > 0$ such that $\alpha - \alpha \upharpoonright m \in A_0^\infty$; thus there exists an integer p such that $\gamma_0(p) = 0$ and $s_p^n \prec \alpha - \alpha \upharpoonright m$. By compactness of n^ω , there are finite sets $F \subseteq \omega \setminus \{0\}$ and $G \subseteq \{p \in \omega / \gamma_0(p) = 0\}$ such that $n^\omega = \bigcup_{m \in F, p \in G} \{\alpha \in n^\omega / s_p^n \prec \alpha - \alpha \upharpoonright m\}$.

We set $A_\gamma := \{s \in n^{<\omega} / \exists p \gamma(p) = 0 \text{ and } s_p^n \prec s\}$ for $\gamma \in 2^\omega$, so that $A_\gamma^\infty \subseteq \mathcal{U}_\gamma^{1, \mathcal{A}}$. Assume that $\gamma(p) = 0$ for each $p \in G$ and let $\alpha \in \mathcal{U}_\gamma^{1, \mathcal{A}}$. Let $p^0 \in \omega$ be such that $\gamma(p^0) = 0$ and $s_{p^0}^n \prec \alpha$. We can find $m_0 > 0$ and $p^1 \in G$ such that $s_{p^1}^n \prec \alpha - \alpha \upharpoonright (|s_{p^0}^n| + m_0)$, and $\alpha \upharpoonright (|s_{p^0}^n| + m_0) \in A_\gamma$. Then we can find $m_1 > 0$ and $p^2 \in G$ such that $s_{p^2}^n \prec \alpha - \alpha \upharpoonright (|s_{p^0}^n| + m_0 + |s_{p^1}^n| + m_1)$, and

$$\alpha \upharpoonright (|s_{p^0}^n| + m_0 + |s_{p^1}^n| + m_1) - \alpha \upharpoonright (|s_{p^0}^n| + m_0) \in A_\gamma.$$

And so on. Thus $\alpha \in A_\gamma^\infty$ and $\{\gamma \in 2^\omega / \forall p \in G \gamma(p) = 0\}$ is a clopen neighborhood of γ_0 and a subset of \mathcal{A}_1 . \square

Proposition 20 \mathcal{A} is $\Sigma_3^1 \setminus D_2(\Sigma_1^0)$ and is co-nowhere dense.

Proof. Let $U := \{\gamma \in 2^\omega / \forall \beta \in 2^\omega \forall \alpha \in n^\omega \exists p [\gamma(p) = 0 \text{ and } s_{(p)_0}^2 \prec \beta \text{ and } s_{(p)_1}^n \prec \alpha]\}$. By compactness of $2^\omega \times n^\omega$, U is a dense open subset of 2^ω . Moreover, if $\gamma \in U$, then $\mathcal{U}_\gamma = \emptyset$, so $U \subseteq \mathcal{A}$ and \mathcal{A} is co-nowhere dense. It remains to see that \mathcal{A} is not open, as in the proof of Proposition 18. As $\mathcal{U}_{1^\infty} = n^\omega$, $1^\infty \in \mathcal{A}$. Let p be an integer satisfying $s_{(p)_0}^2 = \emptyset$ and $s_{(p)_1}^n = 0^q$. We set $\gamma_p(m) := 0$ if and only if $m = p$, and also $P_\infty := \{\alpha \in 2^\omega / \forall r \exists m \geq r \alpha(m) = 1\}$. Then (γ_p) tends to 1^∞ and we have

$$\begin{aligned} \mathcal{U}_{\gamma_p} &= \{\alpha \in n^\omega / \exists \beta \in P_\infty \forall m \ m \neq p \text{ or } s_{(m)_0}^2 \not\prec \beta \text{ or } s_{(m)_1}^n \not\prec \alpha\} \\ &= \{\alpha \in n^\omega / \exists \beta \in P_\infty (\beta, \alpha) \notin 2^\omega \times N_{0^q}\} = \neg N_{0^q}. \end{aligned}$$

So $\gamma_p \notin \mathcal{A}$. \square

5 Ordinal ranks and ω -powers.

Notation. The fact that the ω -powers are Σ_1^1 implies the existence of a co-analytic rank on the complement of A^∞ (see 34.4 in [K1]). We will consider a natural one, defined as follows. We set, for $\alpha \in n^\omega$, $T_A(\alpha) := \{S \in (A^-)^{<\omega} / S^* \prec \alpha\}$. This is a tree on A^- , which is well founded if and only if $\alpha \notin A^\infty$.

The rank of this tree is the announced rank $R_A : \neg A^\infty \rightarrow \omega_1$ (see page 10 in [K1]): we have $R_A(\alpha) := \rho(T_A(\alpha))$. Let $\phi : A^- \rightarrow \omega$ be one-to-one, and $\tilde{\phi}(S) := (\phi[S(0)], \dots, \phi[S(|s| - 1)])$ for $S \in (A^-)^{<\omega}$. This allows us to define the map Φ from the set of trees on A^- into the set of trees on ω , which associates $\{\tilde{\phi}(S)/S \in T\}$ to T . As $\tilde{\phi}$ is one-to-one, Φ is continuous:

$$t \in \Phi(T) \Leftrightarrow t \in \tilde{\phi}[(A^-)^{<\omega}] \text{ and } \tilde{\phi}^{-1}(t) \in T.$$

Moreover, T is well-founded if and only if $\Phi(T)$ is well-founded. Thus, if $\alpha \notin A^\infty$, then we have $\rho(T_A(\alpha)) = \rho(\Phi[T_A(\alpha)])$ because $\tilde{\phi}$ is strictly monotone (see page 10 in [K1]). Thus R_A is a co-analytic rank because the function from n^ω into the set of trees on $\omega^{<\omega}$ which associates $\Phi[T_A(\alpha)]$ to α is continuous, and because the rank of the well-founded trees on ω defines a co-analytic rank (see 34.6 in [K1]). We set

$$R(A) := \sup\{R_A(\alpha)/\alpha \notin A^\infty\}.$$

By the boundedness theorem, A^∞ is Borel if and only if $R(A) < \omega_1$ (see 34.5 and 35.23 in [K1]). We can ask the question of the link between the complexity of A^∞ and the ordinal $R(A)$ when A^∞ is Borel.

Proposition 21 *If $\xi < \omega_1$, $r \in \omega$ and $R(A) = \omega \cdot \xi + r$, then $A^\infty \in \Sigma_{2,\xi+1}^0$.*

Proof. The reader should see [L] for operations on ordinals.

• If $0 < \lambda < \omega_1$ is a limit ordinal, then let (λ_q) be a strictly increasing co-final sequence in λ , with $\lambda_q = \omega \cdot \theta + q$ if $\lambda = \omega \cdot (\theta + 1)$, and $\lambda_q = \omega \cdot \xi_q$ if $\lambda = \omega \cdot \xi$, where (ξ_q) is a strictly increasing co-final sequence in the limit ordinal ξ otherwise. By induction, we define

$$\begin{aligned} E_0 &:= \{\alpha \in n^\omega / \forall s \in A^- \ s \not\prec \alpha\}, \\ E_{\theta+1} &:= \{\alpha \in n^\omega / \forall s \in A^- \ s \not\prec \alpha \text{ or } \alpha - s \in E_\theta\}, \\ E_\lambda &:= \{\alpha \in n^\omega / \forall s \in A^- \ s \not\prec \alpha \text{ or } \exists q \in \omega \ \alpha - s \in E_{\lambda_q}\}. \end{aligned}$$

• Let us show that $E_{\omega \cdot \xi + r} \in \Pi_{2,\xi+1}^0$. We may assume that $\xi \neq 0$ and that $r = 0$. If $\xi = \theta + 1$, then $E_{\lambda_q} \in \Pi_{2,\theta+1}^0$ by induction hypothesis, thus $E_{\omega \cdot \xi + r} \in \Pi_{2,\theta+3}^0 = \Pi_{2,\xi+1}^0$. Otherwise, $E_{\lambda_q} \in \Pi_{2,\xi_q+1}^0$ by induction hypothesis, thus $E_{\omega \cdot \xi + r} \in \Pi_{\xi+1}^0 = \Pi_{2,\xi+1}^0$.

• Let us show that if $\alpha \in A^\infty$, then $\alpha \notin E_{\omega \cdot \xi + r}$. If $\xi = r = 0$, it is clear. If $r = m + 1$ and $s \in A^-$ satisfies $s \prec \alpha$ and $\alpha - s \in A^\infty$, then we have $\alpha - s \notin E_{\omega \cdot \xi + m}$ by induction hypothesis, thus $\alpha \notin E_{\omega \cdot \xi + r}$. If $r = 0$ and $s \in A^-$ satisfies $s \prec \alpha$ and $\alpha - s \in A^\infty$, then we have $\alpha - s \notin E_{\lambda_q}$ for each integer q , by induction hypothesis, thus $\alpha \notin E_{\omega \cdot \xi + r}$.

• Let $s \in A^-$ such that $s \prec \alpha \notin A^\infty$. We have

$$\begin{aligned} \rho(T_A(\alpha - s)) &= \sup\{\rho_{T_A(\alpha-s)}(t) + 1 / t \in T_A(\alpha - s)\} \\ &\leq \sup\{\rho_{T_A(\alpha)}((s)t) + 1 / (s)t \in T_A(\alpha)\} \\ &\leq \rho_{T_A(\alpha)}((s)) + 1 \\ &\leq \rho_{T_A(\alpha)}(\emptyset) < \rho(T_A(\alpha)). \end{aligned}$$

The first inequality comes from the fact that the map from $T_A(\alpha - s)$ into $T_A(\alpha)$, which associates $(s)t$ to t is strictly monotone (see page 10 in [K1]). We have

$$\rho(T_A(\alpha)) \geq [\sup\{\rho(T_A(\alpha - s)) / s \in A^-, s \prec \alpha\}] + 1.$$

Let us show that we actually have equality. We have

$$\rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = \sup\{\rho_{T_A(\alpha)}((s)) + 1 / s \in A^-, s \prec \alpha\} + 1.$$

Therefore, it is enough to notice that if $s \in A^-$ and $s \prec \alpha$, then $\rho_{T_A(\alpha)}((s)) \leq \rho_{T_A(\alpha-s)}(\emptyset)$. But this comes from the fact that the map from $\{S \in T_A(\alpha) / S(0) = s\}$ into $T_A(\alpha - s)$, which associates $S - (s)$ to S , preserves the extension ordering (see page 352 in [K1]).

• Let us show that, if $\alpha \notin A^\infty$, then “ $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$ ” is equivalent to “ $\alpha \in E_{\omega \cdot \xi + r}$ ”. We do it by induction on $\omega \cdot \xi + r$. If $\xi = r = 0$, then it is clear. If $r = m + 1$, then “ $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$ ” is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$ or $\rho(T_A(\alpha - s)) \leq \omega \cdot \xi + m + 1$ ”, by the preceding point. This is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$ or $\alpha - s \in E_{\omega \cdot \xi + m}$ ”, which is equivalent to “ $\alpha \in E_{\omega \cdot \xi + r}$ ”. If $r = 0$, then “ $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$ ” is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$ or there exists an integer q such that $\rho(T_A(\alpha - s)) \leq \lambda_q + 1$ ”. This is equivalent to “ $\forall s \in A^-, s \not\prec \alpha$ or there exists an integer q such that $\alpha - s \in E_{\lambda_q}$ ”, which is equivalent to “ $\alpha \in E_{\omega \cdot \xi + r}$ ”.

• If $\alpha \notin A^\infty$, then $\rho(T_A(\alpha)) \leq \omega \cdot \xi + r + 1$. By the preceding point, $\alpha \in E_{\omega \cdot \xi + r}$. Thus we have $A^\infty = \neg E_{\omega \cdot \xi + r} \in \Sigma_{2 \cdot \xi + 1}^0$. \square

We can find an upper bound for the rank R , for some Borel classes:

Proposition 22 (a) $A^\infty = n^\omega$ if and only if $R(A) = 0$.

(b) If $A^\infty = \emptyset$, then $R(A) = 1$.

(c) If $A^\infty \in \Delta_1^0$, then $R(A) < \omega$, and there exists $A_p \subseteq 2^{<\omega}$ such that $A_p^\infty \in \Delta_1^0$ and $R(A_p) = p$ for each integer p .

(d) If $A^\infty \in \Pi_1^0$, then $R(A) \leq \omega$, and $(A^\infty \notin \Sigma_1^0 \Leftrightarrow R(A) = \omega)$.

Proof. (a) If $\alpha \notin A^\infty$, then $\emptyset \in T_A(\alpha)$ and $\rho(T_A(\alpha)) \geq \rho_{T_A(\alpha)}(\emptyset) + 1 \geq 1$.

(b) We have $T_A(\alpha) = \{\emptyset\}$ for each α , and $\rho(T_A(\alpha)) = \rho_{T_A(\alpha)}(\emptyset) + 1 = 1$.

(c) By compactness, there exists $s_1, \dots, s_p \in n^{<\omega}$ such that $A^\infty = \bigcup_{1 \leq m \leq p} N_{s_m} \in \Delta_1^0$. If $\alpha \notin A^\infty$, then we have $N_{\alpha[\max_{1 \leq m \leq p} |s_m|]} \subseteq \neg A^\infty$, thus $\rho(T_A(\alpha)) \leq \max_{1 \leq m \leq p} |s_m| + 1 < \omega$. So we get the first point. To see the second one, we set $A_0 := 2^{<\omega}$. If $p > 0$, then we set

$$A_p := \{0^2\} \cup \bigcup_{q \leq p} \{s \in 2^{<\omega} / 0^{2q} 1 \prec s\} \cup \{s \in 2^{<\omega} / 0^{2p+1} \prec s\}.$$

Then $A_p^\infty = \bigcup_{q \leq p} N_{0^{2q} 1} \cup N_{0^{2p+1}} \in \Delta_1^0$. If $\alpha_p := 0^{2p-1} 1^\infty$, then $\rho(T_{A_p}(\alpha_p)) = p$. If $\alpha \notin A_p^\infty$, then $\rho(T_{A_p}(\alpha)) \leq p$.

(d) If $A^\infty \in \Pi_1^0$ and $\alpha \notin A^\infty$, then let $s \in n^{<\omega}$ with $\alpha \in N_s \subseteq \neg A^\infty$. Then $\rho(T_A(\alpha)) \leq |s| + 1$. Thus $R(A) \leq \omega$. If $A^\infty \notin \Sigma_1^0$, then we have $R(A) \geq \omega$, by Proposition 21. Thus $R(A) = \omega$. Conversely, we apply (c). \square

Remark. Notice that it is not true that if the Wadge class $< A^\infty >$, having A^∞ as a complete set, is a subclass of $< B^\infty >$, then $R(A) \leq R(B)$. Indeed, for A we take the example A_2 in (c), and for B we take the example for Σ_1^0 that we met in the proof of Proposition 11. If we exchange the roles of A and B , then we see that the converse is also false. This example A for Σ_1^0 shows that Proposition 21 is optimal for $\xi = 0$ since $R(A) = 1$ and $A^\infty \in \Sigma_1^0 \setminus \Pi_1^0$. We can say more: it is not true that if $A^\infty = B^\infty$, then $R(A) \leq R(B)$. We use again (c): we take $A := A_2$ and $B := A \setminus \{0^2\}$. We have $A^\infty = B^\infty = A_2^\infty$, $R(A) = 2$ and $R(B) = 1$.

Proposition 23 *For each $\xi < \omega_1$, there exists $A_\xi \subseteq 2^{<\omega}$ with $A_\xi^\infty \in \Sigma_1^0$ and $R(A_\xi) \geq \xi$.*

Proof. We use the notation in the proof of Theorem 15. Let $T \in \mathcal{T}$, and $\varphi : T \rightarrow T_{\Phi'(T)}(\alpha_0)$ defined by the formula $\varphi(s) := (\phi(s \upharpoonright 0), \dots, \phi(s \upharpoonright |s| - 1))$. Then φ is strictly monotone. If $T \in WF$, then $\alpha_0 \notin (\Phi'(T))^\infty$ and $T_{\Phi'(T)}(\alpha_0) \in WF$. In this case, $\rho(T) \leq \rho(T_{\Phi'(T)}(\alpha_0)) = R_{\Phi'(T)}(\alpha_0)$ (see page 10 in [K1]). Let $T_\xi \in WF$ be a tree with rank at least ξ (see 34.5 and 34.6 in [K1]). We set $A_\xi := \Phi'(T_\xi)$. It is clear that A_ξ is what we were looking for. \square

Remark. Let $\psi : 2^{n^{<\omega}} \rightarrow \{\text{Trees on } n^{<\omega}\}$ defined by $\psi(A) := T_A(\alpha_0)$, and $r : \neg I_{\alpha_0} \rightarrow \omega_1$ defined by $r(A) := \rho(T_A(\alpha_0))$. Then ψ is continuous, thus r is a Π_1^1 -rank on

$$\psi^{-1}(\{\text{Well-founded trees on } n^{<\omega}\}) = \neg I_{\alpha_0}.$$

By the boundedness theorem, the rank r and R are not bounded on $\neg I_{\alpha_0}$. Proposition 23 specifies this result. It shows that R is not bounded on $\Sigma_1^0 \setminus I_{\alpha_0}$.

6 The extension ordering.

Proposition 24 *We equip A with the extension ordering.*

(a) *If $A \subseteq n^{<\omega}$ is an antichain, then A^∞ is in $\{\emptyset\} \cup \{n^\omega\} \cup [\Pi_1^0 \setminus \Sigma_1^0] \cup [\Pi_2^0(A) \setminus \Sigma_2^0]$, and any of these cases is possible.*

(b) *If $A \subseteq n^{<\omega}$ has finite antichains, then $A^\infty \in \Pi_2^0$ (and is not Σ_2^0 in general).*

Proof. Let $G := \{\alpha \in n^\omega \mid \forall r \exists m \exists p \geq r \ \alpha \upharpoonright m \in [(A^-)^p]^*\}$. Then $G \in \Pi_2^0(A)$ and contains A^∞ . Conversely, if $\alpha \in G$, then we have $T_A(\alpha) \cap (A^-)^p \neq \emptyset$ for each integer p , thus $T_A(\alpha)$ is infinite.

(a) If A is an antichain, then each sequence in $T_A(\alpha)$ has at most one extension in this tree adding one to the length. Thus $T_A(\alpha)$ is finite splitting. This implies that $T_A(\alpha)$ has an infinite branch if $\alpha \in G$, by König's lemma. Therefore $A^\infty = G \in \Pi_2^0(A)$.

- If we take $A := \emptyset$, then A is an antichain and $A^\infty = \emptyset$.
- If we take $A := \{(0), \dots, (n-1)\}$, then A is an antichain and $A^\infty = n^\omega$.
- If $A^\infty \notin \{\emptyset, n^\omega\}$, then $A^\infty \notin \Sigma_1^0$. Indeed, let $\alpha_0 \notin A^\infty$ and $s_0 \in A^-$. By uniqueness of the decomposition into words of A^- , the sequence $(s_0^n \alpha_0)_n \subseteq n^\omega \setminus A^\infty$ tends to $s_0^\infty \in A^\infty$.
- If we take $A := \{(0)\}$, then A is an antichain and $A^\infty = \{0^\infty\} \in \Pi_1^0 \setminus \Sigma_1^0$.
- If A is finite, then A^∞ is $\Pi_1^0 \setminus \Sigma_1^0$ or is in $\{\emptyset, n^\omega\}$, by the facts above and Proposition 2.
- If A is infinite, then $A^\infty \notin \Sigma_2^0$ because the map c in the proof of Proposition 2 is an homeomorphism and $(A^-)^\omega$ is not K_σ .
- If $A := \{0^k 1 / k \in \omega\}$, then A is an antichain and $A^\infty = P_\infty$, which is $\Pi_2^0 \setminus \Sigma_2^0$.

(b) The intersection of P_∞ with N_1 can be made with the chain $\{10^k / k \in \omega\}$. So let us assume that A has finite antichains.

• Let us show that A is the union of a finite set and of a finite union of infinite subsets of sets of the form $A_{\alpha_m} := \{s \in n^{<\omega} / s \prec \alpha_m\}$. Let us enumerate $A := \{s_r / r \in \omega\}$. We construct a sequence (A_m) , finite or not, of subsets of A . We do it by induction on r , to decide in which set A_m the sequence s_r is. First, $s_0 \in A_0$. Assume that s_0, \dots, s_r have been put into A_0, \dots, A_{p_r} , with $p_r \leq r$ and $A_m \cap \{s_0, \dots, s_r\} \neq \emptyset$ if $m \leq p_r$. We choose $m \leq p_r$ minimal such that s_{r+1} is compatible with all the sequences in $A_m \cap \{s_0, \dots, s_r\}$, we put s_{r+1} into A_m and we set $p_{r+1} := p_r$ if possible. Otherwise, we put s_{r+1} into A_{p_r+1} and we set $p_{r+1} := p_r + 1$.

Let us show that there are only finitely many infinite A_m 's. If A_m is infinite, then there exists a unique sequence $\alpha_m \in n^\omega$ such that $A_m \subseteq A_{\alpha_m}$. Let us argue by contradiction: there exists an infinite sequence $(m_q)_q$ such that A_{m_q} is infinite. Let t_0 be the common beginning of the α_{m_q} 's. There exists $\varepsilon_0 \in n$ such that $N_{t_0 \varepsilon_0} \cap \{\alpha_{m_q} / q \in \omega\}$ is infinite. We choose a sequence u_0 in A extending $t_0 \mu_0$, where $\mu_0 \neq \varepsilon_0$. Then we do it again: let $t_0 \varepsilon_0 t_1$ be the common beginning of the elements of $N_{t_0 \varepsilon_0} \cap \{\alpha_{m_q} / q \in \omega\}$. There exists $\varepsilon_1 \in n$ such that $N_{t_0 \varepsilon_0 t_1 \varepsilon_1} \cap \{\alpha_{m_q} / q \in \omega\}$ is infinite. We choose a sequence u_1 in A extending $t_0 \varepsilon_0 t_1 \mu_1$, where $\mu_1 \neq \varepsilon_1$. The sequence (u_i) is an infinite antichain in A . But this is absurd. Now let us choose the longest sequence in each nonempty finite A_m ; this gives an antichain in A and the result.

• Now let $\alpha \in G$. There are two cases. Either for each m and for each integer k , $\alpha \upharpoonright k \notin [A^{<\omega}]^*$ or $\alpha - \alpha \upharpoonright k \neq \alpha_m$. In this case, $T_A(\alpha)$ is finite splitting. As $T_A(\alpha)$ is infinite, $T_A(\alpha)$ has an infinite branch witnessing that $\alpha \in A^\infty$, by König's lemma. Otherwise, $\alpha \in \bigcup_{s \in [A^{<\omega}]^*, m} \{s \alpha_m\}$, which is countable. Thus $G \setminus A^\infty \in \Sigma_2^0$ and $A^\infty = G \setminus (G \setminus A^\infty) \in \Pi_2^0$. \square

7 Examples.

• We have seen examples of subsets A of $2^{<\omega}$ such that A^∞ is complete for the classes $\{\emptyset\}$, $\{n^\omega\}$, Δ_1^0 , Σ_1^0 , Π_1^0 , Π_2^0 and Σ_1^1 . We will give some more examples, for some classes of Borel sets. Notice that to show that a set in such a non self-dual class is complete, it is enough to show that it is true (see 21.E, 22.10 and 22.26 in [K1]).

• For the class $\Sigma_1^0 \oplus \Pi_1^0 := \{(U \cap O) \cup (F \setminus O) \mid U \in \Sigma_1^0, O \in \Delta_1^0, F \in \Pi_1^0\}$, we can take $A := \{s \in 2^{<\omega} / 0^2 1 \prec s \text{ or } s = 0^2 \text{ or } \exists p \in \omega \ 10^p 1 \prec s\}$, since $A^\infty = \{0^\infty\} \cup \bigcup_q N_{0^2 q + 2_1} \cup N_1 \setminus \{10^\infty\}$.

• For the class $\check{D}_2(\Sigma_1^0) := \{U \cup F \mid U \in \Sigma_1^0, F \in \Pi_1^0\}$, we can take Example 9 in [St2]: $A := \{s \in 2^{<\omega} / 0 \prec s \text{ or } \exists q \in \omega \ (101)^q 1^3 \prec s \text{ or } s = 10^2\}$. We have

$$A^\infty = \bigcup_{p \in \omega} [N_{(10^2)^{p0}} \cup (\bigcup_{q \in \omega} N_{(10^2)^p (101)^q 1^3})] \cup \{(10^2)^\infty\},$$

which is a $\neg D_2(\Sigma_1^0)$ set. Towards a contradiction, assume that A^∞ is $D_2(\Sigma_1^0)$:

$$A^\infty = U_1 \cap F = U \cup F_2,$$

where the U 's are open and the F 's are closed. Let O be a clopen set separating $\neg U_1$ from F_2 (see 22.C in [K1]). Then $A^\infty = (U \cap O) \cup (F \setminus O)$ would be in $\Sigma_1^0 \oplus \Pi_1^0$. If $(10^2)^\infty \in O$, then we would have $N_{(10^2)^{p0}} \subseteq O$ for some integer p_0 . But the sequence $((10^2)^p (1^2 0)^\infty)_{p \geq p_0} \subseteq O \setminus U$ and tends to $(10^2)^\infty$, which is absurd. If $(10^2)^\infty \notin O$, then we would have $N_{(10^2)^{q0}} \subseteq \neg O$ for some integer q_0 . But the sequence $((10^2)^{q_0} (101)^q 1^\infty)_{q \geq q_0} \subseteq F \setminus O$ and tends to $(10^2)^{q_0} (101)^\infty$, which is absurd.

• For the class $D_2(\Sigma_1^0)$, we can take $A := [A_1^{<\omega}]^* \setminus [A_0^{<\omega}]^*$, where $A_0 := \{010, 01^2\}$ and

$$A_1 := \{010, 01^2, 0^2, 0^3, 10^2, 1^2 0, 10^3, 1^2 0^2\}.$$

We have $A^\infty = A_1^\infty \setminus A_0^\infty$. Indeed, as $A \subseteq [A_1^{<\omega}]^*$, we have $A^\infty \subseteq A_1^\infty$. If $\alpha \in A_0^\infty$, then its decomposition into words of A_1 is unique and made of words in A_0 . Thus $\alpha \notin A^\infty$ and

$$A^\infty \subseteq A_1^\infty \setminus A_0^\infty.$$

Conversely, if $\alpha = a_0 a_1 \dots \in A_1^\infty \setminus A_0^\infty$, with $a_i \in A_1^-$, then there are two cases. Either there are infinitely many indexes i (say i_0, i_1, \dots) such that $a_{i_j} \notin A_0$. In this case, the words $a_0 \dots a_{i_0}, a_{i_0+1} \dots a_{i_1}, \dots$, are in A and $\alpha \in A^\infty$. Or there exists a maximal index i such that $a_i \notin A_0$. In this case, $a_0 \dots a_i 0, 10^2, 1^2 0 \in A$, thus $\alpha \in A^\infty = A_1^\infty \setminus A_0^\infty$. Proposition 2 shows that $A \in D_2(\Sigma_1^0)$. If $A^\infty = U \cup F$, with $U \in \Sigma_1^0$ and $F \in \Pi_1^0$, then we have $U = \emptyset$ because A_1^∞ is nowhere dense (every sequence in A_1 contains 0, thus the sequences in A_1^∞ have infinitely many 0's). Thus A^∞ would be closed. But this contradicts the fact that $((01^2)^n 0^\infty)_n \subseteq A^\infty$ and tends to $(01^2)^\infty \notin A^\infty$. Thus A^∞ is a true $D_2(\Sigma_1^0)$ set.

• For the class $\check{D}_3(\Sigma_1^0)$, we can take $A := ([A_2^{<\omega}]^* \setminus [A_1^{<\omega}]^*) \cup [A_0^{<\omega}]^*$, where $A_0 := \{0^2\}$, $A_1 := \{0^2, 01\}$, and $A_2 := \{0^2, 01, 10, 10^2\}$. We have $A^\infty = (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty$. Indeed, as $A \subseteq [A_2^{<\omega}]^*$, we have $A^\infty \subseteq A_2^\infty$. If $\alpha \in A_1^\infty$, then its decomposition into words of A_2^- is unique and made of words in A_1 . If moreover $\alpha \notin A_0^\infty$, then it is clear that $\alpha \notin A^\infty$ and

$$A^\infty \subseteq (A_2^\infty \setminus A_1^\infty) \cup A_0^\infty.$$

Conversely, it is clear that $A_0^\infty \subseteq A^\infty$. If $\alpha = a_0 a_1 \dots \in A_2^\infty \setminus A_1^\infty$, then the argument above still works. We have to check that $s := a_0 \dots a_{i_0} \notin [A_1^{<\omega}]^*$. It is clear if $a_{i_0} = 10$. Otherwise, $a_{i_0} = 10^2$ and we argue by contradiction.

The length of s is even and the decomposition of s into words of A_1 is unique. It finishes with 0^2 , and the even coordinates of the sequence s are 0. Therefore, $a_{i_0-1} = 0^2$ or 10 ; we have the same thing with a_{i_0-2} , a_{i_0-3} , \dots . Because of the parity, some 0 remains at the beginning. But this is absurd. Now we have to check that $a_0 \dots a_i 0 \notin [A_1^{<\omega}]^*$. It is clear if $a_i = 10^2$. Otherwise, $a_i = 10$ and the argument above works.

Finally, we have to check that if $\gamma \in A_1^\infty$, then $\gamma - (0) \in A^\infty$. There is a sequence p_0, p_1, \dots , finite or not, such that $\gamma = (0^{2p_0})(01)(0^{2p_1})(01) \dots 0^\infty$. Therefore

$$\gamma - (0) = (0^{2p_0}10)(0^{2p_1}10) \dots (0^2)^\infty \in A^\infty.$$

If we set $U_i := \neg A_{2-i}^\infty$, then we see that $A^\infty \in \check{D}_3(\Sigma_1^0)$. If α finishes with 1^∞ , then $\alpha \notin A_2^\infty$; thus A_2^∞ is nowhere dense, just like A^∞ . Thus if $A^\infty = (U_2 \setminus U_1) \cup U_0$ with U_i open, then $U_0 = \emptyset$. By uniqueness of the decomposition of a sentence in A_i^∞ into words of A_{i+1} , we see that A_i^∞ is nowhere dense in A_{i+1}^∞ . So let $x_\emptyset \in A_0^\infty$, $(x_n) \subseteq A_1^\infty \setminus A_0^\infty$ converging to x_\emptyset , and $(x_{n,m})_m \subseteq A_2^\infty \setminus A_1^\infty$ converging to x_n . Then $x_{n,m} \in U_1$, which is absurd. Thus $A^\infty \notin D_3(\Sigma_1^0)$.

• For the class $\check{D}_2(\Sigma_2^0)$, we can take $A := \{s \in 2^{<\omega} / 1^2 \prec s \text{ or } s = (0)\}$. We can write

$$A^\infty = (\{0^\infty\} \cup \bigcup_p N_{0^p 1^2}) \cap (P_f \cup \{\alpha \in 2^\omega / \forall n \exists m \geq n \alpha(m) = \alpha(m+1) = 1\}).$$

Then $A^\infty \notin D_2(\Sigma_2^0)$, otherwise $A^\infty \cap N_{1^2} \in D_2(\Sigma_2^0)$ and would be a comeager subset of N_{1^2} . We could find $s \in 2^{<\omega}$ with even length such that $A^\infty \cap N_{1^2 s} \in \Pi_2^0$. We define a continuous function $f : 2^\omega \rightarrow 2^\omega$ by formulas $f(\alpha)(2n) := \alpha(n)$ if $n > \frac{|s|+1}{2}$, $(1^2 s)(2n)$ otherwise, and $f(\alpha)(2n+1) := 0$ if $n > \frac{|s|}{2}$, $(1^2 s)(2n+1)$ otherwise. It reduces P_f to $A^\infty \cap N_{1^2 s}$, which is absurd.

Summary of the complexity results in this paper:

	Baire category	complexity $\xi = 1$	$\xi = 2$	$\xi \geq 3$
Σ_0	nowhere dense	$\Pi_1^0 \setminus \Sigma_1^0$		
Π_0	co-nowhere dense	$\Sigma_1^0 \setminus \Pi_1^0$		
Δ_1	co-nowhere dense	$K_\sigma \setminus \Pi_2^0$		
Σ_ξ	co-nowhere dense	$\Pi_1^1 \setminus \Delta_1^1$		$\Pi_1^1 \setminus \Delta_1^1$
Π_ξ	co-nowhere dense	$\Pi_1^1 \setminus \Pi_2^0$	$\Pi_1^1 \setminus \Delta_1^1$	$\Pi_1^1 \setminus \Delta_1^1$
Δ	co-nowhere dense	$\Pi_1^1 \setminus \Delta_1^1$		
Σ_ξ	co-nowhere dense	$\Delta_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus \Pi_2^0$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
Π_ξ	co-nowhere dense	$\Delta_2^1 \setminus \Pi_2^0$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
Δ	co-nowhere dense	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$		
$\mathcal{G}_\xi (\xi \in \omega)$		$\Pi_1^0 \setminus \Sigma_1^0$ nowhere dense	$\check{D}_\omega(\Sigma_1^0) \setminus D_\omega(\Sigma_1^0)$	$\Pi_1^1 \setminus D_\omega(\Sigma_1^0)$
\mathcal{F}		$\Pi_1^1 \setminus \Pi_2^0$		
\mathcal{A}_ξ	co-meager	$\check{D}_2(\Sigma_1^0) \setminus D_2(\Sigma_1^0)$ co-nowhere dense	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
\mathcal{M}_ξ	co-meager	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$ co-nowhere dense	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$	$\Sigma_2^1 \setminus D_2(\Sigma_1^0)$
\mathcal{B}		Σ_2^1		
\mathcal{A}	co-nowhere dense	$\Sigma_3^1 \setminus D_2(\Sigma_1^0)$		

8 References.

- [F1] O. Finkel, *Borel hierarchy and omega context free languages*, Theoret. Comput. Sci. 290, 3 (2003), 1385-1405
- [F2] O. Finkel, *Topological properties of omega context free languages*, Theoret. Comput. Sci. 262 (2001), 669-697
- [K1] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1995
- [K2] A. S. Kechris, *On the concept of Π_1^1 -completeness*, Proc. A.M.S. 125, 6 (1997), 1811-1814
- [L] A. Levy, *Basic set theory*, Springer-Verlag, 1979
- [Lo] M. Lothaire, *Algebraic combinatorics on words*, Cambridge University Press, 2002
- [Lou] A. Louveau, *A separation theorem for Σ_1^1 sets*, Trans. A. M. S. 260 (1980), 363-378
- [M] Y. N. Moschovakis, *Descriptive set theory*, North-Holland, 1980
- [S] P. Simonnet, *Automates et théorie descriptive*, Ph. D. Thesis, Université Paris 7, 1992
- [St1] L. Staiger, *ω -languages*, Handbook of Formal Languages, Vol 3, edited by G. Rozenberg and A. Salomaa, Springer-Verlag, 1997
- [St2] L. Staiger, *On ω -power languages*, New Trends in Formal Languages, Control, Cooperation and Combinatorics, Lect. Notes in Comput. Sci. 1218 Springer-Verlag (1997), 377-393